Notation

K - a field i.e. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q$ ← finite field

$\mathbb{Q}(t)$ = field of rational functions

Algebraic closures with $\overline{K}, \overline{\mathbb{R}}, \overline{\mathbb{Q}}$

Polynomial rings

$K[x_1, \ldots, x_n] =$ Poly ring in $x_1, \ldots, x_n$ over K

A lot of the algebraic constructions work over R = a commutative ring with a unit $1 \in R$

Unless stated otherwise $R = K[x_1, \ldots, x_n]$

$K[x_1, \ldots, x_n]$ - infinite dimensional v. space over K with basis of monomials

$$X^a = x_1^{a_1} \cdots x_n^{a_n} \qquad a \in \mathbb{N}^n$$

Every polynomial is:

$$f = \sum_a \underset{\in K}{c_a} X^a$$

The degree of f = $\deg(f) = \max |a| = a_1 + \cdots + a_n$ s.t. $c_a \neq 0$.

Ex]

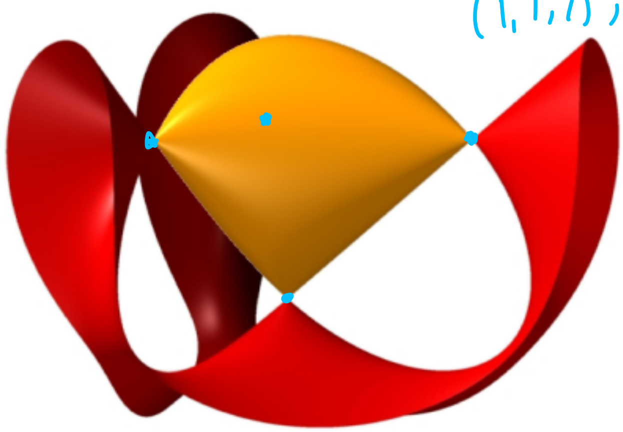
$$f = \det \underbrace{\begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}}_{M} = 2xyz - x^2 - y^2 - z^2 + 1$$

The zero set of f

$$= V(f) = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0 \}$$

The points where M drops rank.

$(1,1,1)$, $(1,-1,-1)$
$(-1,1,-1)$, $(-1,-1,1)$



f has 4 singular points, that is points where

$$\frac{df}{dx} = \frac{df}{dy} = \frac{df}{dz} = 0, \quad f = 0$$

These points are where M has $\operatorname{rank}(M) = 1$.

<u>Ideal</u> in R is a (non-empty) subset $I \subseteq R$ s.t.

$$f \in R,\ g \in I \Rightarrow fg \in I$$

$$f, g \in I \Rightarrow f + g \in I.$$

<u>Def 2</u>] $I \subseteq R$ is an ideal iff $\exists$ a ring hom.

$$\phi : R \to S \quad \text{s.t.} \quad \ker(\phi) = \phi^{-1}(0) = I.$$

S ↑ some com. rings

Prop if $I \subset R$ is an ideal R/I is a ring

Notation Given a subset $F \subseteq R$

$\langle F \rangle$ = smallest ideal containing F

in $R = K[x_1, \dots, x_n]$, $f_1, \dots, f_m \in R$

$$\langle f_1, \dots, f_m \rangle = \left\{ \sum h_i f_i \;\middle|\; h_i \in K[x_1, \dots, x_n] \right\}$$

Fact/Def I, J are ideals in R, the following are ideals

$\underbrace{I + J}_{\text{geo} = \text{intersection}}$, the $\underbrace{I \cap J}_{\text{geo} = \text{union}}$

the Quotient/colon

$$I : J = \langle f \in R \mid f \cdot J \subseteq I \rangle$$

$$IJ = \langle fg \mid f \in I, g \in J \rangle$$

$K[x]$ is a Principal ideal domain (PID)

$\Updownarrow$

every ideal is generated by one element.

in $K[x_1, \dots, x_n]$ every polynomial can be uniquely factored

so $K[x_1, \dots, x_n]$ is a Unique Factorization Domain UFD

However $K[x_1, \dots, x_n]$ is not a PID

$\langle x_1, x_2 \rangle$ is not Principal

Def $f \neq 0$ in a ring R is called

- a <u>nilpotent</u> if $f^m = 0$ for some $m \in \mathbb{N}$
- a <u>zero divisor</u> if $\exists\, g \neq 0 \in R$ s.t. $gf = 0$

R is an <u>integral domain</u> if it has no zero divisors and $1 \neq 0$ i.e. $\{0\}$ is a ring, but <u>not</u> an integral domain

Let I be an ideal in R, we have the following

Property	Def.	R/I
I is maximal	no proper ideal contains I	is a field
I is prime	$f \cdot g \in I \Rightarrow f \in I$ $g \in I$	is a integral domain
I is radical	$(\exists s \in \mathbb{N}$ s.t. $f^s \in I) \Rightarrow f \in I$	has no nilpotent elements
I is primary	$fg \in I$ and $g \notin I \Rightarrow (\exists s \in \mathbb{N} : f^s \in I)$	all zero divisors are nilpotent.

Maximal, prime, primary ideals are proper

Ex $I = \langle x^2 + 10x + 34,\ 3y - 2x - 13 \rangle \subseteq \mathbb{R}[x,y]$ is maximal since

$$\mathbb{C} \cong \mathbb{R}[x,y]/I$$
$$i \longmapsto \frac{1}{13}(x + 5y)$$

Proposition We have the following implications $I \subseteq R$

I is maximal $\Rightarrow$ I is prime $\Rightarrow$ I is radical
$\Rightarrow$ I is primary.

Further I radical and primary $\Rightarrow$ prime.
Every intersection of prime ideals is radical.

Proof If I is maximal $\Rightarrow$ R/I is a field $\therefore$ R/I has no zero divisors.

Suppose I is prime, show I is radical.

i.e. show $f^s \in I \Rightarrow f \in I$

Use induction

$f \in I \Rightarrow f \in I \quad s=1$

now by induction $f^{s-1} \in I \Rightarrow f \in I$

set $g = f^{s-1}$

If $f^s = g \cdot f \in I \Rightarrow$ either $f \in I$

or $g \in I$, $g = f^{s-1}$

Show radical + primary $\Rightarrow$ prime

Let I radical primary

Suppose $f \cdot g \in I$ and $f \notin I$

Since I is primary $\Rightarrow g^s \in I$ for some s

but I is radical $\therefore g \in I \therefore I$ is prime

∎

The Arrows in the Proposition above are not reversible

In $R = \mathbb{R}[x,y]$

- $I = \langle x^2 \rangle$ is primary, but not radical

  (R/I has nil potent elements i.e. $x \in R/I$ is nilpotent)

- $I = \langle x(x-1) \rangle$ is radical but not primary

  ($x, x-1$ are zero divisors in R/I but are not nilpotent)

- $I = \langle x \rangle$ is prime but not maximal

  $R/I \simeq \mathbb{R}[y]$ which is not a feild.

Monday	Tuesday	wednesday	Thursday	Friday
5-6	5-6	1-2	1-2 pm 2-3 pm	