$S_<(I)$ = Set of all monomials $x^b \notin in_<(I)$

standard monomials of I w.r.t $<$

Thm The set $S_<(I)$ of standard monomials is a basis for the k-vector space $k[x_1,..,x_n]/I = R/I$.

Proof The image of $S_<(I)$ in $k[x_1,..,x_n]/I$ is linearly independent since if $f \neq 0$, and $f = 0$ in R/I $\Rightarrow f \in I$

$\Rightarrow f$ has at least $in_<(f)$ not in $S_<(I)$

$\therefore$ any poly $\sum c_\alpha x^\alpha$ s.t. $x^\alpha \in S_<(I)$

cannot be zero in R/I.

$\therefore$ $x^\alpha \in S_<(I)$ are lin. ind.

Now prove $S_<(I)$ spans R/I. Suppose they do not. $\Rightarrow \exists\, x^c \neq \sum c_\alpha x^\alpha$ modulo I for x^α in $S_<(I)$

Assume x^c is the minimal such monomial w.r.t $<$

by assumption $x^c \notin S_<(I) \Rightarrow x^c \in in_<(I)$

$\Rightarrow \exists\, h \in I$ st $in_<(h) = x^c$

Each other monomial in h is smaller w.r.t. $<$ and $\therefore$ is in the k-span of $S_<(I)$ mod I

and $h = 0$ mod I

$\hookrightarrow h = x^c +$ terms in k-span of $S_<(I)$ $= 0$ mod I

x^c is in the k-span of $S_c(I)$ mod I

which is a contradiction $\blacksquare$

Def (Hilbert Function). Let $I \subset k[x_1, \dots, x_n]$ be a monomial ideal

The <u>Hilbert Function</u> h_I takes $\mathbb{N} \to \mathbb{N}$

$h_I(q)$ = the number of monomials of degree q <u>not</u> belonging to I.

= # standard monomials of degree $= q$.

Def (Hilbert series) Let $I \subseteq k[x_1, \dots, x_n]$ be a <u>monomial ideal</u>

Fix a formal variable z. The Hilbert series is

$$HS_I(z) = \sum_{q=0}^{\infty} h_I(q) z^q$$

Begin with the ideal $I = \{0\}$, i.e. count all monomials in $k[x_1, \dots, x_n]$ of a fixed degree

$$HS_{\{0\}}(z) = \frac{1}{(1-z)^n} = \sum_{q=0}^{\infty} \binom{n+q-1}{n-1} z^q$$

Proof uses stars and bars

Number of monomials of degree q in n variables $= \binom{n+q-1}{n-1}$

Note that $\binom{n+q-1}{n-1} = \dfrac{(q+1)\cdots(q+n-1)}{(n-1)!}$ is a polynomial in q.

Ex) Let $I = \langle x_1^{a_1} \cdots x_n^{a_n} \rangle$, $\sum a_i = e$

To count monomials of degree q that are not in I

$\Rightarrow$ Count all monomials $-$ monomials in I

monomials in I of degree q $= \binom{n+q-e-1}{n-1}$

So

$$HS_I(z) = \frac{1-z^e}{(1-z)^n} = \sum_{q=0}^{\infty} \Big[\underbrace{\binom{n+q-1}{n-1} - \binom{n+q-e-1}{n-1}}_{h_I(q)} \Big] z^q$$

For $q \geq e$

$$h_I(q) = \binom{n+q-1}{n-1} - \binom{n+q-e-1}{n-1}$$

is a polynomial in q of degree $n-2$

Ex) Fix an ideal $I = \langle m_1, m_2 \rangle$ in $k[x_1,\dots,x_n]$

where $\deg(m_i) = e_i$

Count monomials of degree q by

0) Find all monomials of degree q

1) Find all monomials of degree q divisible by m_1

2) add all monomials of degree q divisible by m_2

3) Subtract # monomials divisible by both m_1 and m_2 $e_{12} = \deg(\operatorname{lcm}(m_1, m_2))$

$$HS_I(z) = \frac{1 - z^{e_1} - z^{e_2} + z^{e_{12}}}{(1-z)^n}$$

the Hilbert function

$$\binom{n+q-1}{n-1} - \binom{n+q-e_1-1}{n-1} - \binom{n+q-e_2-1}{n-1} + \binom{n+q-e_{12}-1}{n-1}$$

Thm The Hilbert series of a monomial ideals $I \subseteq K[x_1, \dots, x_n]$ is

$$HS_I(z) = \frac{K_I(z)}{(1-z)^n}$$

where $K_I(z)$ is a polynomial with integer coefficients and $K_I(0) = 1$. Further $\exists$ a polynomial, HP, called the Hilbert polynomial, of I s.t. $HP(q) = h_I(q)$ for sufficiently large q.

Proof sketch we will count monomials via inclusion-exclusion.

Let $m_1, \dots, m_r$ be monomials which minimally generate I

For any subset τ of the index set $\{1, 2, \dots, r\}$

$m_\tau = \operatorname{lcm}\{m_i \mid i \in \tau\}$, $e_\tau = \deg(m_\tau)$

$m_\phi = 1$, $e_\phi = 0$

Using inclusion exclusion we write

$$HS_I(z) = \frac{\underbrace{\sum_{\tau \subseteq \{1,2,\dots,r\}} (-1)^{|\tau|} z^{e_\tau}}_{= K_I(z)}}{(1-z)^n}$$

$K_I(0) = 1$

If we regrouped terms similar to the examples

$$h_I(q) = \underbrace{\sum_{\tau \subseteq \{1,\dots,r\}} (-1)^{|\tau|} \binom{n+q-e_\tau-1}{n-1}}_{\text{is a polynomial for } q \text{ large enough}} = HP_I(q)$$

for large enough q

is a polynomial for q large enough
i.e. $q \geq \deg(\mathrm{lcm}(m_1, \dots, m_r))$

Def (Dimension and Degree). Let I be a monomial ideal and write

$$HP_I(q) = \frac{g}{(d-1)!} q^{d-1} + \text{lower terms}$$

If $HP \neq 0$ the <u>dimension of I</u> is defined as $\underline{d}$

and the <u>degree of I</u> is defined as $\underline{g}$ — can show that g is a non-negative integer.

If $HP_I(q) = 0$ we say I is zero dimensional

$\to$ $K[x_1,\dots,x_n]/I$ is a finite dimensional K-vector space

Then $\deg(I) = \dim_K \left(K[x_1,\dots,x_n]/I \right)$

Now consider an arbitrary ideal I in $K[x_1,\dots,x_n]$

Let $<$ be any degree-compatible monomial order

i.e. s.t.

$|a| < |b|$ implies $x^a < x^b$

$\sum a_i$

(so Grevlex, Glex, not lex)

Lemma The number of standard monomials of I in degree q is independent of the choice of monomial ordering provided the order is degree compatible.

Proof Let $R = k[x_1, \dots, x_n]$ and let $R_{\leq q}$ denote the vector space of polynomials of degree $\leq q$

write $I_{\leq q} = I \cap k[x_1, \dots, x_n]_{\leq q}$

$\uparrow$ subspace of $R_{\leq q}$ which is in I

$$S_<(I)_{\leq q} = S_<(I) \cap R_{\leq q}$$

Claim: $S_<(I)_{\leq q}$ is a k-vector space basis for

$$R_{\leq q} / I_{\leq q}$$

know $S_<(I)_{\leq q}$ is linearly ind. by previous result.

Show $S_<(I)_{\leq q}$ spans $R_{\leq q}/I_{\leq q}$

Let $f \in R$, if we find $f \bmod I$ via the division algorithm, using a GB of I,

then $f \bmod I$ = remainder after division $= r$

$\Rightarrow$ r is a k-linear combination of monomials which are divisible by any monomial in $<(I)$

Since our order is degree compatible

$in_<(r) \leq in_<(f) \quad \Rightarrow \quad \underbrace{\text{degree}(r) \leq \deg(f)}$

from division alg applied to f

degree compatible order

So the remainder will be in span of $S_<(I)_{\leq q}$. Since $R_{\leq q}/I_{\leq q}$ doesn't depend on $<$ then neither does $|S_<(I)_{\leq q}|$ ∎

<u>Def</u>/ Let I be any ideal in $R = k[x_1, \dots, x_n]$

The function $AfH_I : \mathbb{N} \to \mathbb{N}$

$$q \mapsto \dim_k\left(k[x_1, \dots, x_n]_{\leq q} / I_{\leq q}\right)$$

is called the <u>affine Hilbert function</u>

$R = k[x_1, \dots, x_n]$

The hilbert function h_I of I is defined to be hilbert function $in_<(I)$ with <u>some degree compatible order</u>

$$h_I(q) := h_{in_<(I)}(q) = |S_<(I)_{\leq q}| - |S_<(I)_{\leq q-1}|$$

$$= \dim_k(R_{\leq q}/I_{\leq q}) - \dim_k(R_{\leq q-1}/I_{q-1})$$

The affine Hilbert function can be obtained from hilbert func (and vice versa)

$$AfH(q) = \sum_{j=0}^{q} h_I(j$$