Varieties

Let k be a field, $f_1, \dots, f_s \in k[x_1, \dots, x_n]$

$$V(f_1, \dots, f_s) = \{ p = (p_1, \dots, p_n) \in k^n \mid f_1(p) = \cdots = f_s(p) = 0 \}$$

is the affine variety associated to $f_1, \dots, f_s$

Note that for any $f \in I = \langle f_1, \dots, f_s \rangle$ and any $p \in V(f_1, \dots, f_s)$

$$f(p) = 0$$

By Hilbert basis theorem all ideals have a finite gen set, so we can define:

$$V(I) := V(f_1, \dots, f_s) \quad \text{for } \underline{\text{any}} \text{ generating set } I = \langle f_1, \dots, f_s \rangle.$$

Note: two distinct ideals can define the same variety

Consider f_1, f_2 non-constant homogeneous polys.

then $\langle f_1^2, f_2^5 \rangle \subsetneq \langle f_1, f_2 \rangle = \langle f_1, f_1 + f_2 \rangle$

but $V(f_1^2, f_2^5) = V(f_1, f_2) = V(f_1, f_1 + f_2)$

Ch. 6

The Nullstellensätz will address the question of what ideals correspond (1-1) to varieties for fields that are algebraically closed or real closed (ie $k = \mathbb{C}$ or $k = \mathbb{R}$)

In the real case $V_{\mathbb{R}}(f_1^2, f_2^5) = V_{\mathbb{R}}(f_1^2 + f_2^5)$

Def A variety $V(I)$ is called irreducible if it cannot be written as a finite union of proper subvarieties of k^n, i.e.

$V(I)$ is irreducible iff for any ideals $J, J' \subseteq k[x_1, \dots, x_n]$

if $V(I) = V(J) \cup V(J') \Rightarrow$ Either $V(I) = V(J)$
or $V(I) = V(J')$

we will see that all varieties can be decomposed into irreducibles. (in M2 "decompose")
$\downarrow$
· primary-decomposition

Zariski Topology

we turn k^n into a topological space (for any field k) using the Zariski topology

- The closed sets are all varieties $V(I)$ for some ideal $I \subseteq k[x_1, \dots, x_n]$
- The open sets are complements of closed sets
- If $k = \mathbb{R}$, $k = \mathbb{C}$ we also have the Euclidean topology on k^n, which has many more open sets

Ideal which vanishes on a point set

Consider the maximal ideal $m = \langle x_1 - p_1, \dots, x_n - p_n \rangle \subseteq k[x_1, \dots, x_n]$

The point $(p_1, \dots, p_n)$ is in $V(I)$ iff

$$I \subseteq m$$

For any subset $W \subseteq K^n$ we define the Ideal of W as

$$I(W) := \{ f \in K[x_1, \dots, x_n] \mid f(p) = 0 \ \forall p \in W \}$$

This is a radical ideal (if $f^r \in I(W)$ $\wedge$ $f^r(p) = 0$ $\forall p \in W$ $\Rightarrow f(p) = 0 \Rightarrow f \in I(W)$)

all polynomials which vanish on W.

Suggests 1-1 correspondence between radical ideals and varieties

Thm) The set W is a variety iff $W = V(I(W))$.

Further, given any two varieties $W, U \subseteq K^n$

$$U \subseteq W \text{ iff } I(W) \subseteq I(U).$$

Lemma) For ideals $J \subset I$ in $K[x_1, \dots, x_n]$ we have $V(I) \subseteq V(J)$.

Prop) A variety $W \subseteq K^n$ is irreducible iff its ideal $I(W)$ is prime.

Proof) Suppose that $I(W)$ is prime but $W = V(J) \cup V(J')$

If $W \not\subseteq V(J) \Rightarrow \exists f \in J$, $v \in W$ s.t. $f(v) \neq 0$

$\Rightarrow f \notin I(W)$

For any $g \in J'$, fg vanishes on $V(J)$ ($f \in J$), and vanishes on $V(J')$ ($g \in J'$)

$\therefore fg \in I(W)$

$\Rightarrow g \in I(W)$ (Since $I(W)$ is prime and $f \notin I(W)$)

$\Rightarrow J' \subseteq I(W)$ $\therefore$ (By Lemma) $V(I(W)) \subseteq V(J')$

and since W is a variety

$$W = V(I(W))$$

$\therefore \quad W \subseteq V(J')$

and by def $V(J') \subseteq W$

$\therefore \quad W = V(J')$.

Now suppose W is irreducible, and $f \cdot g \in I(W)$

Show $f \in I(W)$ or $g \in I(W)$

Since $f \cdot g \in I(W)$

$$\Rightarrow \quad W = W \cap V(fg) = W \cap (V(f) \cup V(g))$$
$$= (W \cap V(f)) \cup (W \cap V(g))$$

Since W is irreducible

$\Rightarrow$ Either $W = W \cap V(f)$ OR $W = W \cap V(g)$

Say $W = W \cap V(f) \Rightarrow W \subseteq V(f)$

$\Rightarrow f(p) = 0 \ \forall p \in W$

$\Rightarrow f \in I(W)$

$\therefore I(W)$ is prime. ■

Basic objects are irreducible varietes and prime ideals.

Def The <u>spectrum</u> of the ring R is the set of all proper prime ideals

$$\operatorname{spec}(R) :\simeq \{ P \subsetneq R \mid P \text{ is a prime ideal} \}$$

The set $\operatorname{spec}(R)$ is a topological space with the Zariski topology.

The closed sets are the <u>varietes</u>

$$V(I) = \{ p \in \operatorname{spec}(R) \mid I \subseteq p \}$$

$\operatorname{spec}(R)$ records all prime ideals and how they geometrically.

Consider $R = k[x_1, \dots, x_n]$

zero set space

$V(R) = k^n$

The topological $\mathrm{Spec}(R)$ contains

- The usual points $(p_1, \dots, p_n) \in k^n$, represented by maximal ideals $\langle x_1 - p_1, \dots, x_n - p_n \rangle$
- points corresponding to all irreducible subvarieties of k^n (not just the dim 0 ones)
- k^n itself

Ex] $\mathrm{Spec}(\mathbb{R}[x])$ contains points such as $\langle x-a \rangle$ $a \in \mathbb{R}$ but also $\langle x^2+1 \rangle$

$\uparrow$ $x^2 = -1$ has no real solutions.

Def] For a variety $W \subseteq k^n$ the coordinate ring of W to be $R = k[W] := k[x_1, \dots, x_n] / I(W)$

set of equivalence classes

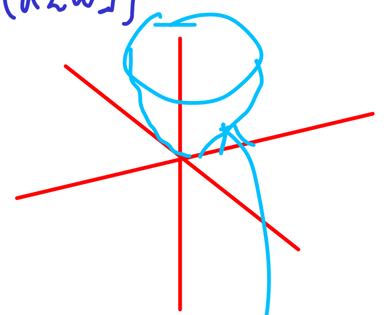
Thm] $k[W] \cong \{$ polynomial functions $f : W \to k \}$

Since elements of $J = I(W)$ vanish on W evaluating $f \in k[W]$ at a point in W will not depend on the representing f (since we work mod J)

Thm] There is a bijective correspondence between prime ideals in $k[W]$ and prime ideals in $k[x_1, \dots, x_n]$ which contain $I(W)$.

Geometrically these prime ideals correspond to irreducible subvarieties of W.

Amoung these ideals, which are points in $\mathrm{Spec}(K[W])$, are points $(p_1, \dots, p_n) \in W$ represented by maximal ideals $\langle x_1 - p_1, \dots, x_n - p_n \rangle$ in $\mathrm{Spec}(K[W])$



Ex) A parabaloid in $\mathbb{R}^3$

Its ideal is $J = \langle z - x^2 - y^2 \rangle$

with coordinate ring is $K[x,y,z]/J$

The Zariski top. on $\mathrm{Spec}(\mathbb{R}[x,y,z]/J)$ has

- The classical points on the real surface
- Pairs of complex congugate points satisfying $z = x^2 + y^2$
- points for all irriducible curves on the surface

  i.e. $\langle z^2+1, z - x^2 - y^2 \rangle$

  $\uparrow$ which has no real points

<u>Geometric version of Chinese Reminder Thm</u>

Fix $n_1, \dots, n_\ell \in \mathbb{Z}$ which are pairwise coprime

In terms of varieties (in $\mathrm{Spec}(\mathbb{Z})$)

The fact that $\langle n_i \rangle + \langle n_j \rangle = \mathbb{Z}$

is equivalent to the associated varieties having an empty intersection

$$V(n_i) \cap V(n_j) = \emptyset$$

For each n_i we are giving $a_i \in \mathbb{Z}/n_i\mathbb{Z}$

i.e. a function on the variety $V(n_i)$

Since the varieties do not intersect we expect a unique function on

$V(n_1) \cup \cdots \cup V(n_\ell)$ that restricts to a_i on each $V(n_i)$

The union of varieties corresponds to intersection of ideals

$$\langle n_1 \rangle \cap \cdots \cap \langle n_\ell \rangle = \langle \prod n_i \rangle = \langle N \rangle \qquad \prod n_i = N$$

This is the Chinese Remainder Thm, i.e $\exists$ a unique

$a \in \mathbb{Z}/N\mathbb{Z}$ s.t. $a = a_i \mod n_i$.

<u>Def</u> A polynomial map $F : k^n \longrightarrow k^m$ is a map

$$x \longmapsto (F_1(x), \ldots, F_m(x))$$

for F_i polynomials.

For varieties W_1, W_2 a map $f : W_1 \longrightarrow W_2$ is a morphism if it is the restriction of a polynomial map from $k^m \longmapsto k^n$ (where $W_1 \subseteq k^n$, $W_2 \subseteq k^m$).

Given a morphism $f : W_1 \longrightarrow W_2$ the <u>Pull back</u> is

$$f^* : k[W_2] \longrightarrow k[W_1]$$
$$: g \longmapsto g \circ f$$

Homework

will show that f^* induces a continuous map of topological spaces $\quad \mathrm{Spec}(K[W_1]) \longrightarrow \mathrm{Spec}(K[W_2])$

Ex] Consider the parametrization map

$$f: \mathbb{R} \longrightarrow V(y-x^2) \subseteq \mathbb{R}^2$$
$$\lambda \longmapsto (\lambda, \lambda^2)$$

The pull back is a ring isomorphism

$$f^*: \mathbb{R}[x,y]/\langle y-x^2\rangle \longrightarrow \mathbb{R}[z]$$
$$x \longmapsto z$$
$$y \longmapsto z^2$$

i.e. $\mathbb{R} \cong V(y-x^2)$

Ex] Consider $I_1 = \langle x^2 - y^2 \rangle$, $I_2 = \langle x^2 - 2y^2 \rangle$ and $I_3 = \langle x^2 + y^2 \rangle$ in $K[x,y]$.

- $x^2 - y^2 = (x-y)(x+y)$ over any field, so I_1 is never prime $\therefore V(I_1)$ is never irreducible

- $x^2 - 2y^2$ factors over $\mathbb{R}$ and $\mathbb{C}$, but not over $\mathbb{Q}$
  $\therefore I_2$ is prime in $\mathbb{Q}$, but not over $\mathbb{R}$, or $\mathbb{C}$

- $x^2 + y^2$ factors over $\mathbb{C}$, but not over $\mathbb{R}$ or $\mathbb{Q}$

  in $\mathbb{C}$ $\quad V(x^2+y^2) = V(x+iy) \cup V(x-iy)$

  in $\mathbb{R}, \mathbb{Q}$ $\quad (0,0)$ is the only point in $V(x^2+y^2)$ and $\langle x^2+y^2 \rangle$ is prime.

Def $A \subseteq k^n$ is called <u>constructible</u> if it can be described as a finite union of (set) differences of varieties.

A subset $B \subseteq \mathbb{R}^n$ is called <u>semi-algebraic</u> if it can be described as the solution to a finite system of polynomial inequalities (both $\geq, >$)

Every constructable set (in $\mathbb{R}^n$) is semi-algebraic, but not the other way around.

Ex work in $\mathbb{R}^2$, $\mathbb{R}[x,y]$, $V(x,y) = \{(0,0)\}$

$\mathbb{R}^2 - \{0,0\} = V(0) - V(x,y)$

is constructable

$\mathbb{R}^2_{\geq 0} = \{(u,v) \in \mathbb{R}^2 \mid u \geq 0, v \geq 0\}$ is semi-algebraic

but not constructable since $\mathbb{R}^2_{\geq 0}$ is closed (but not a variety)

Geometric Dimension and Degree

Ex Let V be a linear subspace of k^n

$\dim(V) =$ dim of V as a linear space (i.e. vector space)

$$\text{so} \quad k[V] \cong k[\underbrace{x_{s+1}, \ldots, x_n}_{\dim V \text{ coordinates for } V}] \cong k[x_1, \ldots, x_n] / \langle x_1, \ldots, x_s \rangle$$

when $s = n - \dim(V)$

e.g. if $\dim(V) = n-1 \Rightarrow V$ is the intersection of k^n with a hyperplane.

Def | Define $\dim(V) = \dim(I(V)) = \dim(\mathrm{in}_<(I(V)))$ for all affine varieties $V \subseteq k^n$.

Thm | If $V_1 \subsetneq V_2$ are affine varieties then $\dim(V_1) \leq \dim(V_2)$. If V_1, V_2 are irreducible then $\dim(V_1) < \dim(V_2)$.

A method to compute $\dim(V(I))$

Note that $V(\mathrm{in}_<(I)) =$ union of linear spaces in k^n

- compute a G.B. of I, to obtain $\mathrm{in}_<(I)$
- Suppose $\mathrm{in}_<(I) = \langle m_1, \dots, m_\ell \rangle$; find the smallest (w.r.t. cardinality) set of variables $S = \{x_{i_1}, \dots, x_{i_d}\}$ s.t. every m_j is divisible by one of the variables in S.
- $n-d = \dim(V(\mathrm{in}_<(I))) = \dim(V(I))$

Geometric meaning of degree (k is algebraically closed)

Def 1 | For an affine variety $V = V(I) \subseteq k^n$

$$\deg(V) = \deg(I) = \deg(\mathrm{in}_<(I))$$

Def 2 | Let $V \subseteq k^n$ be an affine variety, let $L \subseteq k^n$ be a general linear space with $\dim(V) + \dim(L) = n$

Then $\deg(V) = \#(V \cap L)$ # of points, which is finite,

A general linear polynomial ℓ is not a zero divisor in $k[V]$

$\therefore$ the Hilbert function of $I + \langle \ell \rangle$ is changed s.t.
the dimension drops by one and the degree stays the same

• If $\dim(I) = 0$, I is radical,
$\deg(I) = \# V(I)$.

Thm (Bezout's Theorem) Let $f_1, \dots, f_\ell$ be general polynomials in n variables of degrees $d_1, \dots, d_\ell \geq 0$.

$\deg(V(f_1) \cap \cdots \cap V(f_\ell)) = \deg(f_1) \cdots \deg(f_\ell)$

For $I = \langle f_1, \dots, f_\ell \rangle$ we have $\dim(I) = n - \ell$
$\deg(I) = d_1 \cdots d_\ell$

if k is alg. closed
$\Rightarrow \dim(V(I)) = n - \ell$
$\operatorname{degree}(V(I)) = d_1 \cdots d_\ell$.

Remark For any $f_1, \dots, f_\ell$ $\dim(\langle f_1, \dots, f_\ell \rangle) \geq n - \ell$,
when k is alg closed $\dim(V(f_1, \dots, f_\ell)) \geq n - \ell$
and when $= n$ we call $V(f_1, \dots, f_\ell)$ a complete intersection.

A polynomial $f = \sum_{i=1}^{m} c_{a_i} x^{a_i}$ is <u>general</u> for all
$c = (c_{a_1}, \dots, c_{a_m}) \in U \subseteq k^m$ where U is Zariski dense in k^m

A set U is Zariski dense in k^m is U is open in the Zariski topology.