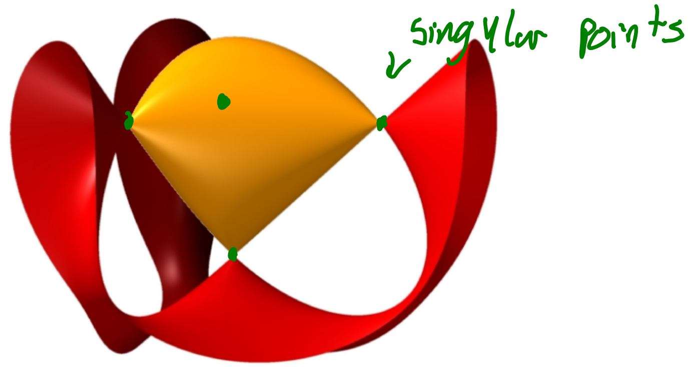
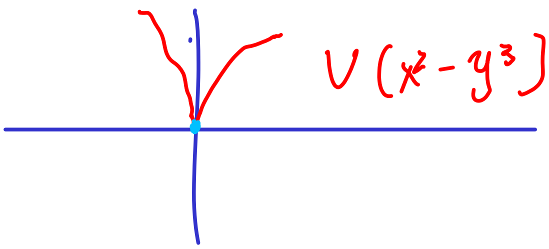
Singularities





$V(x^2-y^3)$

$J = Jac_{(x,y)}(x^2-y^2) = [2x, 3y^2]$

↑ for generic x, y

$Rank(J(x,y)) = 1$

but $Rank(J(0,0)) = 0$

More generally, for $f \in k[x_1, \dots, x_n]$ a point $p \in V(f)$ is <u>singular</u> if <u>all</u> partial derivatives vanish at p,

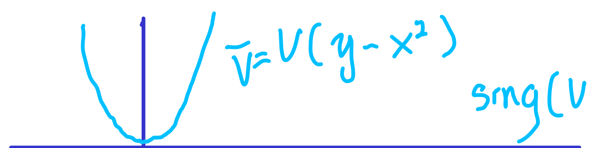
i.e. $Jac(f)(p) = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right] = [0, \dots, 0]$

The <u>singular locus</u> of $V(f)$ is

$$Sing(V(f)) := V\left(f, \frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_1}{\partial x_n}\right)$$

If $sing(V(f)) = \phi$ we say $V(f)$ is smooth

e.g.



$V = V(y-x^2)$

$sing(V) = V(y-x^2, -2x, 1)$
$= V(1) = \phi$

Smoothness lets us approximate $V(f)$ by the tangent vector $\left(\frac{df}{dx_1}(p), \dots, \frac{df}{dx_n}(p)\right)$ in a (small) neighborhood of p

Suppose $I = \langle f_1, \ldots, f_r \rangle \subseteq k[x_1, \ldots, x_n]$ is a prime ideal defining an irreducible variety $Y = V(I) \subseteq k^n$

Let $\mathrm{Jac}_x(f_1, \ldots, f_r)(p) := \begin{bmatrix} \frac{df_1}{dx_1}(p) & \cdots & \frac{df_1}{dx_n}(p) \\ \vdots & & \\ \frac{df_r(p)}{dx_1} & \cdots & \frac{df_r(p)}{dx_n} \end{bmatrix}$

A point $p \in Y$ is <u>singular</u> iff

$$\mathrm{rank}(\mathrm{Jac}_x(f_1, \ldots, f_r)(p)) < n - \dim(Y) = \mathrm{codim}(Y)$$

If p is not singular we say p is <u>smooth</u>

$$\mathrm{rank}(\mathrm{Jac}_x(f_1, \ldots, f_r)(p)) = n - \dim(Y) = \mathrm{codim}(Y).$$

The set of all singular points is:

$$\mathrm{Sing}(Y) = V(I + J)$$

singular locus

J = ideal defined by all $(\mathrm{codim}(Y)) \times (\mathrm{codim}(Y))$ minors of $\mathrm{Jac}_x(f_1, \ldots, f_r)$

(Informally)

The tangent space to Y at p, T_pY, is the vector space spanned by all possible tangent vectors to Y at p

The kernel of $\mathrm{Jac}_x(f_1, \ldots, f_r)(p)$ is a vector space parallel to the tangent space

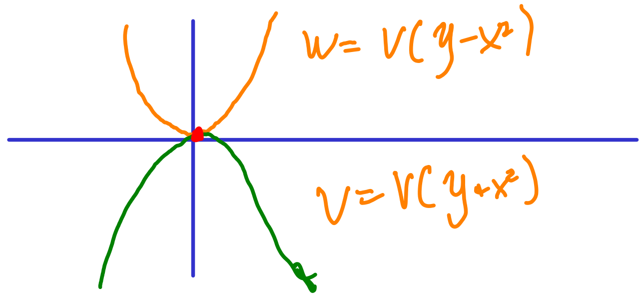
Since, if p is smooth, $\dim \ker \operatorname{Jac}(f_1, \dots, f_r)(p) = n - (n - \dim Y)$
$= \dim Y$

$\Rightarrow \quad \dim Y = \dim T_p Y$

Note

If $X = W \cup V$ and $p \in W$, $p \in V$

$\Rightarrow$ p is a singular point of X



$X = W \cup V$

$(0,0)$ is singular

$X = V((y - x^2)(y + x^2))$
$= V(y^2 - x^4)$

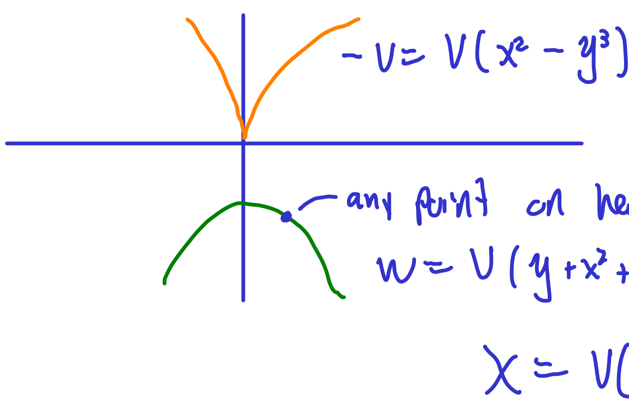
$\operatorname{Jac}(y^2 - x^4) = [-4x^3, 2y]$

On the other hand

if $X = W_1 \cup \dots \cup W_r$

and $p \in W_i$, $p \notin W_j$ $i \neq j$

then p is singular in X iff p is singular in W_i



any point on here is smooth on $W = V(y + x^2 + 1)$ and on $X = W \cup V$

$X = V((y + x^2 + 1)(x^2 - y^3))$

$\operatorname{Sing}(X) = \{(0,0)\}$

(Bertini's Theorem)

Let $I = (f_1, \dots, f_r) \subseteq k[x_1, \dots, x_n]$ with f_i general polynomials of degree d_i. Then $\mathrm{codim}(V(I)) = r$.

If $r < n$ then I is prime, and $V(I)$ is smooth (and irreducible)

Projective Varieties

Let V be a vector space of dimension $n+1$ (e.g. k^{n+1})

The points in $\mathbb{P}(V)$ ← Projective space are line through the origin in V

$[a_0 : \cdots : a_n] \in \mathbb{P}(V)$ denotes a line

$$L = \{\lambda(a_0, \dots, a_n) \mid \lambda \neq 0 \in k\} \text{ in } V$$

Def

$$\mathbb{P}(V) := (V - \{0\}) / \sim$$
$$= \left\{ [v] \;\middle|\; \begin{array}{l} v \in V - \{0\} \text{ and } [v_1] \sim [v_2] \text{ iff} \\ v_1 = \lambda v_2 \text{ for some } \lambda \in k^* = k - \{0\} \end{array} \right\}$$

$\mathbb{P}(V)$ is compact in the classical topology.

Consider the Zariski open set

$$S_i = \{[a_0 : \cdots : a_n] \in \mathbb{P}(V) \mid a_i \neq 0\}$$

check that the map $\varphi : k^n \longrightarrow S_i$

$$(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \longmapsto [a_0 : \cdots : 1 : \cdots a_n]$$

is a 1-1 correspondence.

$$\mathbb{P}(V) = \bigcup_{i=0}^{n} S_i$$ ≃ a union of affine patches

Define $\mathbb{P}^n := \mathbb{P}(V)$

Zero sets (i.e. varieties) in $\mathbb{P}^n$ (projective varieties)

Q: For a poly $f \in k[x_0, \dots, x_n]$ how do we "evaluate" f at an equivalence class

i.e. an arbitrary polynomial might vanish at one representative, but not another.

A: Use homogeneous polynomials. A polynomial

$$f = \sum_{i=1}^{m} c_i x^{b_i} \quad \text{is homogeneous}$$

$$\text{if } d = \deg(x^{b_i}) = \deg(x^{b_j}) \quad \forall i, j$$

$$\deg(x^{b_i}) = b_i^{(0)} + \dots + b_i^{(n)} = d$$

e.g. $x^2 y + y x^2 + xyz$ is homogeneous

$x^2 + y$ is not

Suppose f is homogeneous of degree d in $k[x_0, \dots, x_n]$ it now makes sense to evaluate f at $[a_0 : \dots : a_n]$ since

$d = b_i^{(0)} + \dots + b_i^{(n)}$

$$f(\lambda a_0, \dots, \lambda a_n) = \sum c_i (\lambda a_0)^{b_i^{(0)}} \cdots (\lambda a_n)^{b_i^{(n)}}$$

$$= \sum c_i \lambda^d a^{b_i} = \lambda^d \sum c_i a^{b_i} = \lambda^d f(a_0, \dots, a_n)$$

$\therefore$ if f vanishes at a representative $a = [a_0 : \dots : a_n]$

it vanishes at all other representatives of a.

Let $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ be homogeneous polynomials.

we can now define a projective variety as

$$V(f_1, \dots, f_r) = \{ [a_0 : \dots : a_n] \in \mathbb{P}^n \mid f_1(a_0, \dots, a_n) = \dots = f_r(a_0, \dots, a_n) = 0 \}$$

An ideal $I = \langle f_1, \dots, f_r \rangle$ is called homogeneous

if it is generated by homogeneous polynomials

i.e. if $f_1, \dots, f_r$ to be homogeneous

Define

$$V(I) := V(f_1, \dots, f_r)$$

Note: I contains many non-homogeneous polynomials

i.e. if $\deg(f_1) \neq \deg(f_2)$ $f_1 + f_2$ is not homogeneous.

Irrelevant Ideal

Note that $0 \notin \mathbb{P}^n$

$$\therefore V(x_0, \dots, x_n) = \emptyset$$

So we call the ideal $\langle x_0, \dots, x_n \rangle$ the irrelevant ideal.

Since

$$V(I) \cup V(x_0, \dots, x_n) = V(I \cdot (x_0, \dots, x_n)) = V(I)$$

For $X = V(I) \subseteq \mathbb{P}^n$ a projective variety, the affine cone

$\hat{X}$ over X is the affine variety $V(I) \subset k^{n+1}$,

i.e. the zero set of the same eqs in k^{n+1}

Note a point in X becomes a line (through the origin) in k^{n+1}

Def | $X \subseteq \mathbb{P}^n$

$$\dim(X) := \dim(\hat{X}) - 1 \quad , \quad \deg(X) := \deg(\hat{X})$$

i.e. we define degree and dimension of X to be those of $\hat{X}$.

Note $\dim(\mathbb{P}^n) = \dim(V(0) \subseteq k^{n+1}) - 1 = n+1-1$
$= n$

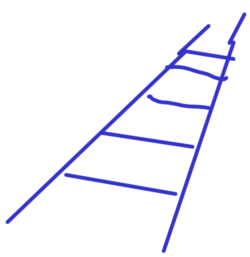
Projective space is nice

E.g. parallel lines in k^2 do not intersect

e.g. $V(y-x-1) \cap V(y-x) = \phi$

But <u>any</u> two lines in $\mathbb{P}^2$ intersect

$$V(y-x-z) \cap V(y-x) = \{[1:1:0]\}$$



two parallel lines meet "at infinity"

Note: If X is any projective variety (of degree ≥ 2) then the affine cone $\hat{X}$ is <u>always</u> singular at the point $0 \in k^{n+1}$, however if this is the only singular point of $\hat{X}$ then X is smooth.

Projective closure of an affine variety

Suppose $Y \subseteq k^n$ is an affine variety with $I = I(Y) \subseteq k[x_1, \dots, x_n]$

The <u>projective closure</u> $\overline{Y} \subseteq \mathbb{P}^n$ of Y is defined by the ideal $\overline{I} \subseteq k[x_0, x_1, \dots, x_n]$ generated by the (infinite) set of homogeneous polynomials

homogenization

$$\Big\{ \underbrace{x_0^{\deg(g)} \cdot g\left(\tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}\right)}_{\text{the homogenization of } g = \overline{g}} \;\Big|\; g \in I \Big\}$$

e.g.

$$g = x_1^2 x_2 - x_2^3 - x_1 + 4$$

$$\overline{g} = x_1^2 x_2 - x_2^3 - x_1 x_0^2 + 4 x_0^3$$

The ideal $\overline{I}$ can be computed as follows:

Prop Let $I \subseteq k[x_1, \dots, x_n]$, and let $G = \{g_1, \dots, g_r\}$ be its reduced Gröbner basis (for a degree compatible monomial order)

Then $\overline{I} = \langle \overline{g}_1, \dots, \overline{g}_r \rangle$.

Proof: Let $f = f(x_0, x_1, \dots, x_n)$ be any polynomial in $\overline{I}$

The <u>dehomogenization</u> $f(1, x_1, \dots, x_n) \in I$, hence (via the division alg. dividing $f(1, x_1, \dots, x_n)$ by G) we have

$$f(1, x_1, \dots, x_n) = \sum_{i=1}^{r} h_i(x_1, \dots, x_n)\, g_i(x_1, \dots, x_n)$$

where $\underline{\deg(h_i g_i) \leq \deg(f(1, x_1, \dots, x_n))} = \deg(f)$ requires a degree compatible mon.-order.

Then $f(x_0, \dots, x_n) = \overline{f(1, x_1, \dots, x_n)} = x_0^{\deg(f)} \sum_{i=1}^{r} h_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) g_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$

$$= \sum x_0^{u_i} x_0^{\deg(h_i)} \underline{x_0^{\deg(g_i)}} h_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \underline{g_i\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)}$$

$u_i \geq 0$

$$= \sum x_0^{u_i} \overline{h_i} \, \overline{g_i}$$

$$\in \langle \overline{g}_1, \dots, \overline{g}_r \rangle$$

$$\therefore \; f \in \langle \overline{g}_1, \dots, \overline{g}_r \rangle \quad \forall f \in \overline{I}.$$

<u>Corrollary</u> The dimension and degree of an affine variety $Y \subseteq k^n$ and its projective closure $\overline{Y} \subseteq \mathbb{P}^n$ are the same

$$\dim(\overline{Y}) = \dim(Y) \quad \text{and} \quad \deg(\overline{Y}) = \deg(Y)$$

<u>Proof (sketch)</u>:

Let $I = I(Y)$ and let $G = \{g_1, \dots, g_r\}$ be a G.B. of I.

Note that (Since we use a degree compatible monomial order and we can take $x_0 < x_i$)

$$in_<(g_i) = in_<(\overline{g_i})$$

$$in_<(I) = \langle in_<(g_1), \dots, in_<(g_r) \rangle \subseteq k[x_1, \dots, x_n]$$

$$in_<(\overline{I}) = \langle in_<(g_1), \dots, in_<(g_r) \rangle \subseteq k[x_0, \dots, x_n]$$

↑ x_0 does not apear

↑ as a finite set, i.e. these are generated by the same monomials (but in diffent rings)

Since dimension and degree are determined by $in_<(I)$ one can check that the degree of the hilbert poly of $\langle in_<(G) \rangle$ in $k[x_0, \dots, x_n]$ has degree 1 greater than that of $\langle in_<(G) \rangle$ in $k[x_1, \dots, x_n]$

but the $\dfrac{\text{lead coefficient}}{(\dim(\overline{I})-1)!}$ is the same

i.e. $\deg(\langle in_<(G)\rangle \subseteq k[x_1,\dots,x_n]) = \deg(\langle in_<(G)\rangle \subseteq k[x_0,\dots,x_n])$

$\dim(\langle in_<(G)\rangle \subseteq k[x_0,\dots,x_n]) = \dim(\langle in_<(G)\rangle \subseteq k[x_1,\dots,x_n])+1$

but. $\dim(\overline{Y}) = \dim(\hat{\overline{Y}})-1$

$= \dim(\overline{I})-1$

$= \dim(I)+1-1 = \dim(I)$

$= \dim(Y)$. ∎

Thm Fix an alg. closed field. Let $X, Y \subseteq \mathbb{P}^n$ be projective varieties

Then $\operatorname{codim}(X\cap Y) \leq \operatorname{codim}(X) + \operatorname{codim}(Y)$

In particular if $\operatorname{codim}(X)+\operatorname{codim}(Y) \leq n$

$\Rightarrow X\cap Y \neq \phi$.

Note all hypothesis are needed

i.e. in $\mathbb{C}^3$ two parallel hyperplanes don't intersect

Ex) in $\mathbb{P}^3$ over $\mathbb{R}$

$X = V(x_0^2+x_1^2-x_2^2+x_3^2)$, $Y = V(x_0^2+x_1^2+x_2^2-x_3^2)$

$X\cap Y$ = 2 points over $\mathbb{R}$

$[0:0:1:1]$, $[0:0:1:-1]$