Solving and decomposing

For a system $f_1 = \cdots = f_r = 0$ of polynomials in $K[x_1, \ldots, x_n]$ what does it mean to solve?

What does $X = V(f_1, \ldots, f_r)$ look like

- If $\dim(X) = 0$ - find all points
- If $\dim(x) > 0$ - we want to describe each irreducible component.

Consider $f(x) \in K[x]$, $I = \langle f(x) \rangle$

$\dim(V(I)) = 0$

$$f = \prod_{i=1}^{l} g_i^{a_i}$$

where each a_i is irreducible

So $V(I) = V(g_1) \cup \cdots \cup V(g_l)$

$$I = \underbrace{\langle g_1 \rangle^{a_1}}_{\text{primary ideal}} \cap \cdots \cap \langle g_k \rangle^{a_k}$$

Note the factorization depends on the field K

when $K = \mathbb{C}$ $\quad g_i(x) = x - u_i$

if $K = \mathbb{R}$ $\quad g_i(x)$ is linear or quadratic

if $K = \mathbb{Q}$ $\quad g_i(x)$ can have arbitrary degree with $K[x]/\langle g_i \rangle$ will be the field extension required to represent the roots of $g_i(x)$

Prop Any variety $W \subseteq K^n$ can be uniquely represented as a finite union $W = V_1 \cup \cdots \cup V_r$ where $V_j \not\subseteq V_i$ $i \neq j$ and V_i is irreducible.

Proof First existance. W is either irreducible, or

$$
\begin{aligned}
W = {} & W_1 \cup V_1 \\
& \mid \\
& W_2 \cup V_2 \\
& \quad \mid \\
& \quad W_2 \cup V_2 \text{ etc.}
\end{aligned}
$$

So $W \supseteq W_1 \supseteq W_2 \supseteq W_3 \supseteq \cdots$ which gives an ascending chain of ideals

$$I(W) \subseteq I(W_1) \subseteq I(W_2) \subseteq \cdots$$

By the Hilbert basis thm this chain stabalizes

$\therefore$ we have a finite decomposition.

Now uniquness:

Suppose $W = W_1 \cup \cdots \cup W_s = V_1 \cup \cdots \cup V_r$

Fix a $i_0 \in \{1, \dots, s\}$ since $W_{i_0} = W \cap W_{i_0}$

$$
\Rightarrow \quad
\begin{aligned}
W_{i_0} &= (V_1 \cup \cdots \cup V_r) \cap W_{i_0} \\
&= \bigcup_{i=1}^{r} (V_i \cap W_{i_0})
\end{aligned}
$$

Since W_{i_0} is irreducible $\Rightarrow$ $W_{i_0} = V_{j_0} \cap W_{i_0}$

$$\Rightarrow \quad W_{i_0} \subseteq V_{j_0}$$

By the same argument $V_{j_0} \subseteq W_{i_1}$

but $W_{i_0} \not\subseteq W_{i_1}$ $\forall\, i_0 \neq i_1$

$$\Rightarrow \quad i_1 = i_0$$

$$\Rightarrow \quad W_{i_0} = V_{j_0}$$

$\therefore$ the decompositions are the same ∎

Primary Decomposition

In chapter 6 we will prove that $\exists$ a 1-1 identification between radical (homogeneous-saturated) ideals and affine varieties (Projective varieties).

($I : (x_0, \dots, x_n)^\infty$)

So $V(I) = V_1 \cup \cdots \cup V_r$ (for I radical)

$\Rightarrow \quad I = P_1 \cap \cdots \cap P_r$ when $V_i = V(P_i)$

$P_i = I(V_i)$

What about non-radical ideals?

in $\langle f(x) \rangle = \langle g_1 \rangle^{a_1} \cap \cdots \cap \langle g_r \rangle^{a_r}$

$\uparrow$ 1-variable

E.g. $I = \langle x^2, y \rangle$ is not an intersection of powers of prime ideals

suppose $I = \bigcap_i P_i^{a_i} \Rightarrow P_i \supseteq I \quad \forall i$

But only prime ideal contain $\langle x^2, y \rangle$

is $\langle x, y \rangle$ but $\langle x, y \rangle \neq \langle x^2, y \rangle$

$$(\langle x,y\rangle)^2 = \langle x^2, xy, y^2\rangle \neq \langle x^2, y\rangle$$

I is primary ideal. $\left[\begin{array}{l} ab \in I,\ a \notin I \\ \Rightarrow b^r \in I \text{ for some } r \end{array}\right]$

we will decompose into primary components

Def (Noetherian Ring). A ring R is Noetherian iff every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq \cdots \quad \text{stabalizes}$$

i.e. $\quad I_j = I_{j+1} = \cdots \quad$ for some j.

Def Let I be an ideal in a ring R. I is irreducible iff whenever $I = J_1 \cap J_2$ for some $J_1, J_2 \subseteq R \Rightarrow I = J_1$ or $I = J_2$.

Thm Let I be an ideal in a Noetherian ring R. Then $\exists$ primary ideals $q_1, \dots, q_r$ in R s.t

$$I = q_1 \cap \cdots \cap q_r$$

Proof: First show every ideal in R is a finite intersection of irreducible ideals.

Suppose not, let I_1 be an ideal which is not a finite intersection of irreducible ideals.

$I_1 = I_2 \cap J_2$ and $I_1 \neq I_2$, $J_1 \neq J_2$ and

$I_1 \subsetneq I_2$

$I_1 \subsetneq J_2$

$I_2 = I_3 \cap J_3$

$I_4 \cap J_4$

This gives us an ascending chain
$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$ of strictly ascending ideals

But this is a contradiction of Noetherianness of R.

$\therefore$ I_1 is a finite intersection of irreducible ideals

Now prove that every irreducible ideal q is primary

Now we may replace R with R/q and assume $q = \{0\}$

$\therefore$ suppose $\{0\}$ is irreducible and show $\{0\}$ is a primary ideal

i.e. show that if $\boxed{ab = 0,\ a \neq 0 \Rightarrow b^n = 0}$

$\Updownarrow$ b is nil potent.

Def'n $\operatorname{Ann}(b) := \{x \in R \mid bx = 0\}$

$\uparrow$ Anilator

Consider the ascending chain

$$\operatorname{Ann}(b) \subseteq \operatorname{Ann}(b^2) \subseteq \operatorname{Ann}(b^3) \subseteq \cdots$$

Since R is Noetherian $\Rightarrow$ $\operatorname{Ann}(b^n) = \operatorname{Ann}(b^{n+1})$ for some n

Claim: $\langle a \rangle \cap \langle b^n \rangle = \{0\}$ $\Rightarrow$ This will finish the proof since $\{0\}$ is irreducible $\Rightarrow$ either $a = 0$ or $b^n = 0$ but by assumption $a \neq 0$ $\Rightarrow b^n = 0$ $\therefore \{0\}$ is primary

Proof of claim

Suppose

$$\lambda a = \mu b^n \in \underbrace{\langle a \rangle \cap \langle b^n \rangle}_{\text{arbitrary element of}}$$

(for $\lambda, \mu \in R$)

then

$$\lambda \underbrace{ab}_{=0} = \mu b^{n+1} = 0$$ since $ab = 0$ by assumption

$\Rightarrow \quad u \in Ann(b^{n+1}) = Ann(b^n) \quad \Rightarrow \quad u b^n = 0$

but $u b^n$ is an arbitray element of $\langle a \rangle \cap \langle b^n \rangle$

$\therefore \quad \langle a \rangle \cap \langle b^n \rangle = \{0\}$.

Now we can decompose arbitray ideal into int. of primary ideals. Is this decomposition unique?

Ex| $\langle x^2, xy \rangle = \langle x \rangle \cap \langle x, y \rangle^2 = \langle x \rangle \cap \langle x^2, y \rangle$

A: No, not unique. however if we take radicals of each component in either decomposition we get

$\langle x \rangle, \langle x, y \rangle$

Note that the ideal is primary, but not irreducible

$I = (x^2, xy, y^2)$, $\sqrt{I} = \langle x, y \rangle$

but $I = (x, y^2) \cap (y, x^2)$

From now on assume any primary decomp is irredunt i.e.

$I = q_1 \cap \cdots \cap q_n$ then $\bigcap_{i \neq i_0} q_i \nsubseteq q_{i_0} \quad \forall i_0$

Lemma| Let q be a primary ideal. $p = \sqrt{q}$ is the unique smallest prime ideal containing q $(q \subseteq p)$.

Def| If q is a primary ideal in a Noetherian ring R, $p = \sqrt{q}$ we say q is p-primary.

Lemma If $q_1, \dots, q_\ell$ are p-primary ideals so is $q_1 \cap \cdots \cap q_\ell$. In particular $\sqrt{q_1 \cap \cdots \cap q_\ell} = p$

to see this last part

$$I = q_1 \cap \cdots \cap q_\ell$$

$a \in \sqrt{I}$ iff $a^n \in I$ iff $a^n \in q_i \ \forall i$

iff $a \in \sqrt{q_i} = p$ iff $a \in p$

Idea: Given any primary decomposition

$$I = \underbrace{q_1 \cap q_2 \cap \cdots \cap q_w}_{} \cap \cdots \cap q_r$$

if all $q_1, \dots, q_w$ are p-primary replace them with the primary $\bar{q} = q_1 \cap \cdots \cap q_w$

Def A minimal primary decomposition is

$$I = q_1 \cap q_2 \cap \cdots \cap q_\ell$$

s.t.

- q_i is primary
- $\sqrt{q_i} \neq \sqrt{q_j}$ for $i \neq j$ ← these are unique
- $\bigcap_{j \neq i_0} q_j \not\subseteq q_{i_0} \quad \forall \ 1 \leq i_0 \leq \ell$.

Note $\langle x^2, xy \rangle = \langle x \rangle \cap \langle x, y \rangle^2 = \langle x \rangle \cap \langle x^2, y \rangle$

is an example of 2 minimal primary decompositions.

Def

$a \in R$, I an ideal — colon ideal or ideal quotient

Recall $I : a = \{ b \in R \mid ab \in I \}$

Lemma 1 If p is a prime ideal in a ring R, $q_1, \dots, q_r$ are ideals and

$$p \supseteq q_1 \cap \cdots \cap q_r \Rightarrow p \supseteq q_j \text{ for some } j \in \{1, \dots, r\}$$

and further if q_j is prime $\forall j \Rightarrow p = q_j$ for some j.

Lemma 2 In a Noetherian ring every ideal contains a power of its radical.

Thm For any ideal in a ring R with minimal primary decomposition

$$I = q_1 \cap \cdots \cap q_r$$

then the ideals $p_i = \sqrt{q_i}$ do not depend on the choice of decomposition. Further

$$\{p_1, \dots, p_r\} = \{p \subset R \mid p \text{ is a prime ideal and } p = \sqrt{I:a} \text{ for some } a \in R\}$$

If R is Noetherian

$$\{p_1, \dots, p_r\} = \{p \subset R \mid p = I:a \text{ is a prime ideal for some } a \in R\}.$$

Note if $a \in I$
$I:a = \{b \in R \mid ab \in R\}$
$= R$

Proof of Thm

Outline of Proof:

- Show that if $\sqrt{I:a}$ is prime for some $a \in R$ then $\sqrt{I:a} = \sqrt{q_{i_0}}$ for some i_0
- Show for any q_j $\exists a \in R$ s.t. $\sqrt{I:a} = \sqrt{q_{i_0}}$

From this it follows that

$$\{\sqrt{q_1}, \dots, \sqrt{q_r}\} = \{\sqrt{I:a} \mid \sqrt{I:a} \text{ is prime and } a \in R\}$$

Note this set does not depend on the chosen decomposition.

Fix a decomposition

$I = q_1 \cap \cdots \cap q_r$ then

$I:a = \bigcap_{i=1}^{r} q_i : a = \bigcap_{\substack{i=1 \\ a \notin q_i}}^{r} q_i : a$

$\sqrt{I:a} = \bigcap_{\substack{j=1 \\ a \notin q_j}}^{r} \sqrt{q_j : a}$

Now show $a \notin q_j \Rightarrow \sqrt{q_j : a} = \sqrt{q_j}$

Suppose $b \in \sqrt{q_j : a} \Rightarrow b^n \in q_j : a$
$\Rightarrow ab^n \in q_j$

Since q_j is primary $a \notin q_j \Rightarrow (b^n)^m \in q_j$

$\Rightarrow b \in \sqrt{q_j}$

$\Rightarrow \sqrt{q_j : a} \subseteq \sqrt{q_j}$

and $\sqrt{q_j} \subseteq \sqrt{q_j : a}$ since $q_j \subseteq q_j : a$

$\therefore \sqrt{q_j} = \sqrt{q_j : a}$ each of these is prime

$\therefore \sqrt{I:a} = \bigcap_{a \notin q_j} \sqrt{q_j}$

Now suppose $\sqrt{I:a}$ is prime (since we are interested only in the set of $\sqrt{I:a}$ s.t. $\sqrt{I:a}$ is prime)

$\therefore$ Lemma 1 $\Rightarrow \sqrt{I:a} = \sqrt{q_i}$ for some i

Hence whenever $\sqrt{I:a}$ is prime it is equal to some $p_i = \sqrt{q_i}$.
Now we need only show <u>all</u> $p_1, \ldots, p_r$ arise in this way.

Consider some $\sqrt{q_{i_0}}$. Since the primary decomposition is minimal

$\exists\ a \in \left(\bigcap_{j \neq i_0} q_j \right) - q_{i_0}$ (otherwise $q_{i_0} \subseteq \bigcap_{j \neq i_0} q_j$)

$\Rightarrow$ Since $a \in q_j\ \forall j \neq i_0$ and $a \notin q_{i_0}$

then $\sqrt{I:a} = \bigcap_{q_j \text{ s.t } a \notin q_j} \sqrt{q_j : a} \underset{\substack{\uparrow \\ \text{By above}}}{=} \bigcap_{a \notin q_j} \sqrt{q_j} = \sqrt{q_{i_0}}$

$\therefore$ every $\sqrt{q_{i_0}}$ for all $i_0 \in \{1, \ldots, r\}$ appears in

$$\{ P \subset R \mid P \text{ is a prime ideal and } P = \sqrt{I:a} \text{ for some } a \in R \}$$

and these are the only ideals which appear

$\therefore$

$$\{\sqrt{q_1}, \ldots, \sqrt{q_r}\} = \{ P \subset R \mid P \text{ is a prime ideal and } P = \sqrt{I:a} \text{ for some } a \in R \}$$

$\therefore$ $\uparrow$ is independent of the primary decomp since $\uparrow$ is.

Now suppose R is Noetherian. ....