Def) If $I = q_1 \cap \cdots \cap q_n$ is a minimal primary decomposition, the associated primes are $p_1 = \sqrt{q_1}, \ldots, p_r = \sqrt{q_r}$

By Thm the associated primes are unique.

Geometric meaning $I = q_1 \cap \cdots \cap q_r$ - minimal primary decomposition

$$\sqrt{I} = \sqrt{q_1} \cap \cdots \cap \sqrt{q_r} \quad \text{- prime decomp}$$

($\sqrt{q_1} = p_1$, $\sqrt{q_r} = p_r$)

$$V(I) = V(p_1) \cup \cdots \cup V(p_r) \quad \text{— irreducible decomp}$$

$\therefore$ every component in an irreducible decomp. of $V(I)$ comes from a primary component of I

However we may have $V(p_j) \subseteq V(p_i)$ for some $i \neq j$.

Ex) $I = \langle x^2, xy \rangle = \langle x \rangle \cap \langle x^2, y \rangle$

$\sqrt{I} = \langle x \rangle \cap \langle x, y \rangle = \langle x \rangle$

($\langle x \rangle$: minimal prime; $\langle x, y \rangle$: embedded prime)

$$V(I) = V(x) \cup V(x,y) = V(x)$$

($V(x,y)$: point $(0,0)$; $V(x)$ = y-axis)

$V(x,y)$ is embedded inside $V(x)$

Def) For an ideal $I = q_1 \cap \cdots \cap q_r$ — minimal primary decomp.

Let

$$\mathrm{Ass}(I) = \{p_1, \ldots, p_r\}, \quad p_i = \sqrt{q_i}$$

The elements of $\mathrm{Ass}(I)$ which are minimal (w.r.t inclusion) are the minimal associated primes. The others are the embedded primes.

R = Noetherian Ring

An embedded prime p of $I \subseteq R$ must contain a minimal prime $\tilde{p}$ (by def) i.e. $\tilde{p} \subsetneq p$

$\Rightarrow V(\tilde{p}) \supsetneq V(p)$

i.e. the irreducible comp. $V(\tilde{p})$ of $V(I)$ strictly contains the irreducible variety $V(p)$

Note the minimal primes correspond exactly to the irreducible components of $V(I)$.

In terms of zero sets we don't see the embedded primes, and we can't distinguish prime and primary ideals since $\sqrt{I}$ and I define the same variety

In modern algebraic geometry we define schemes to address this (idea: this will associate a distinct "geometric" object to each distinct ideal)

This begins with the Zariski topology on

$$\text{Spec}(R) = \{ p \subseteq R \mid p \text{ is a prime ideal} \}$$

(R a noetherian ring) if $R = k[x_1, \ldots, x_n]$ Spec(R) ~ general version of k^n

From I an ideal of R the subscheme defined by I is $\text{Spec}(R/I)$

Now $\langle x \rangle$ and $\langle x^2 \rangle$ are different geometric objects

since $\text{Spec}(\mathbb{C}[x]/\langle x \rangle) = \{c\}$, but $\text{Spec}(\mathbb{C}[x]/\langle x^2 \rangle) = \langle x \rangle$

Lemma A prime ideal is a minimal prime of I iff it is a minimal element (w.r.t to inclusion) among primes containing I.

proof Since embedded primes contain a minimal prime (by def) it is enough to show that every prime $p \in R$ containing I also contains an prime in $Ass(I)$.

Suppose $p \supseteq I = \bigcap_{i=1}^{r} q_i$

By Problem #7 $\Rightarrow p \supseteq q_{i_0}$ for some i_0

$\Rightarrow \sqrt{p} = p \supseteq \sqrt{q_{i_0}}$

$\therefore$ p contains an associated prime ∎

Def (saturation)

Let I, J be ideals in a Noetherian ring R. Consider the following chain

$$I : J \subseteq I : J^2 \subseteq \cdots\cdots$$

This chain stabilizes, say at $I : J^m = I : J^{m+1}$

Define $\underbrace{I : J^\infty}$ $= I : J^m$

called the saturation of I wrt J

If $J = \langle f \rangle$ we write $I : f^\infty$

Def 2 (saturation)

$$I : J^\infty = \{ f \in R \mid \forall g \in J, \exists m \geq 0 \text{ s.t } f g^m \in I \}$$

Aside

Theorem If k is algebraically closed field, I, J are ideal in $k[x_1, \dots, x_n]$ then

$$V(I : J^\infty) = \overline{V(I) - V(J)} \quad \leftarrow \text{zariski closure}$$

Back to Primary decomp

we will show that the "geometric parts" of a primary decomposition are unique, i.e. that the primary components corresponding to minimal primes are unique, so when we have dif. primary decomps we are changing embedded components.

Thm Let R be a Noetherian ring, $I = \bigcap_{i=1}^{r} q_i$ be a minimal primary decomposition. The primary ideals q_{i_0} corresponding to a minimal prime $p_{i_0} = \sqrt{q_{i_0}}$ is

$$q_{i_0} = I : (a_1 \cdots a_{i_0 - 1} a_{i_0+1} \cdots a_r)^\infty$$

where $a_j \in p_j - p_{i_0}$ $\quad j = 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, r$.

Note if p_{i_0} were not minimal then a_j would not be defined for all j

$\therefore$ minimality is needed for p_{i_0}.

Proof set $\bar{a} = a_1 \cdots a_{i_0-1} a_{i_0+1} \cdots a_r$ with $a_j \in p_j - p_{i_0}$, $j = 1, 2, \dots, i_0-1, i_0+1, \dots, r$.

$$I : \bar{a}^\infty = \{ f \in R \mid \exists m \text{ s.t. } f \cdot \bar{a}^m \in I \}$$

Show containment both ways. First show $q_{i_0} \subseteq I : \bar{a}^\infty$.

Let $f \in q_{i_0}$

since $\sqrt{q_{i_0}}$ is prime $\bar{a} \notin \sqrt{q_{i_0}}$ (since $\bar{a}$ is a product of things not in $\sqrt{q_{i_0}}$)

But $f \cdot \bar{a} \in q_j$ for $j \neq i_0$ since $\bar{a} \in q_j$

Further $f\bar{a} \in q_{i_0}$ since $f \in q_{i_0}$

$$\Rightarrow f\bar{a} \in I \Rightarrow f \in I : \bar{a}^\infty$$

Now suppose $f \in I : \bar{a}^\infty \Rightarrow f\bar{a}^m \in I$ for some m

$$\Rightarrow f\bar{a}^m \in q_{i0}$$

But q_{i0} is primary, $\therefore$ if $f \notin q_{i0}$

$$\Rightarrow \bar{a} \in \sqrt{q_{i0}} = P_{i0} \quad \text{but } \bar{a} \notin P_{i0}$$

$$\Rightarrow f \in q_{i0}$$

$$\therefore I : \bar{a}^\infty \subseteq q_{i0}$$

$$\therefore q_{i0} = I : \bar{a}^\infty$$

Mapping and projecting (Elimination)

varieties

$$\varphi : X \to Y \subseteq k^m$$

$$x \mapsto (f_1(x), \dots, f_m(x))$$

Q: what does the image $\varphi(X)$ of a polynomial map look like? Can we compute it?

A: It may not be a variety however its closure (the smallest variety containing the image) is a variety and we can compute it.

Fix an alg. closed field k and work in $k[x_1, \dots, x_n]$.

Let $W = V(I)$ for an ideal I

Consider a set $S \subseteq k^n$ (we allow S to be anything)

One can check

$$I(S) = \{ f \in k[x_1, \dots, x_n] \mid f(a) = 0 \ \forall a \in S \}$$

is a radical ideal in $k[x_1, \dots, x_n]$

<u>Def</u>) We say $\bar{S} = V(I(S))$ is the <u>Zariski Closure</u> of S.

<u>Prop</u>) If $S \subseteq k^n$, $\bar{S} = V(I(S))$ is the smallest variety that contains S, i.e. if $W \subseteq k^n$ is an affine variety

that contains S, $S \subseteq W$, $\Rightarrow \overline{S} \subseteq W$.

Proof | If $W \supseteq S \Rightarrow I(W) \subseteq I(S)$ (check this)

but then $V(I(W)) \supseteq V(I(S))$

$$\underset{W}{\parallel} \qquad \underset{\overline{S}}{\parallel}$$

$\therefore \overline{S} \subseteq W$. ∎

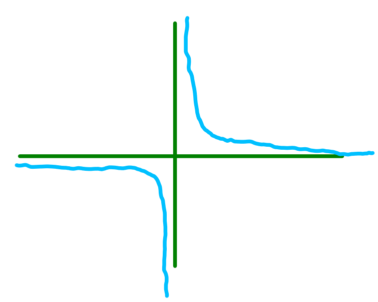
Now let $m < n$ and consider the projection map

$$\pi : k^n \longrightarrow k^m$$

$$(p_1, \dots, p_m, p_{m+1}, \dots, p_n) \longmapsto (p_1, \dots, p_m)$$

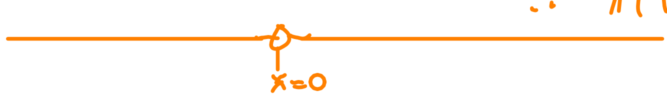
If $W = V(I)$ is a variety in k^n, $\pi(W)$ need not be

Ex] $\pi : k^2 \longrightarrow k$, $W = V(xy - 1)$
$(p_1, p_2) \longmapsto p_1$



$\pi(W) = k - \{0\}$

$\parallel$

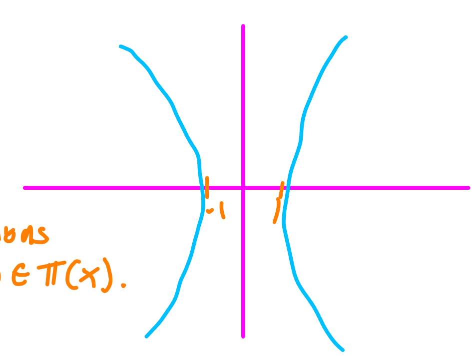


$\therefore \pi(W)$ is not closed

On the other hand if we change coordinates

$x \longmapsto x + y$
$y \longmapsto x - y$

$X = V((x+y)(x-y) - 1)$



$\pi(X) = k$

$\overline{\pi(X)} = \pi(X)$ $\therefore \pi(X)$ is closed

but $x = 0$
$y = \pm i$ are solutions $\therefore 0 \in \pi(X)$.

Thm Let $I \subseteq k[x_1,\dots,x_n]$ (← k-algebraically closed) be an ideal, $W = V(I)$ it's variety.

Let $m<n$, and π the projection $\pi: K^n \to K^m$, $\pi: (p_1,\dots,p_m,p_{m+1},\dots,p_n) \mapsto (p_1,\dots,p_m)$.

Then $\overline{\pi(W)} = V(J)$ is defined by the

elimination ideal

$$J = I \cap k[x_1,\dots,x_m].$$

If I is radical or prime (or homogeneous) (← I won't prove now) then so is J

Proof: If J is not a prime ideal $\Rightarrow \exists\, f,g \in k[x_1,\dots,x_m]$ s.t. $f\cdot g \in J$ but $f \notin J$ and $g \notin J$, but $J \subseteq I$

$\therefore I$ is not prime

If J is not radical $\Rightarrow \exists\, f \in k[x_1,\dots,x_m]$ and $r \geq 2$ s.t. $f^r \in J$, $f \notin J$, but $f^r \in I$ $\overline{\in k[x_1,\dots,x_m]}$

$\Rightarrow f \notin I$

Similarily

$$\sqrt{I} \cap k[x_1,\dots,x_m] = \sqrt{I \cap k[x_1,\dots,x_m]} = \sqrt{J}$$

and since $V(\sqrt{J}) = V(J)$ we may suppose that I and J are radical.

Now show $V(J) = V(I(\pi(W)))$

$$= \overline{\pi(W)}$$

Aside (we need Nullstellensatz) ← chapter 6.

Thm (Nullstellensatz) Let $I \subseteq K[x_1,\dots,x_n]$ be an ideal and let K be algebraically closed. Then

$$I(V(I)) = \sqrt{I}.$$

Now show $V(J) = V(I(\pi(W))) = \overline{\pi(W)}$

First show $\overline{\pi(W)} \subseteq V(J)$.

Let $f \in J$. If $a = (a_1, \dots, a_n) \in W = V(I)$

$\Rightarrow f(a) = 0$ since $f \in I$

$f \in K[x_1, \dots, x_m]$ So we have

$$f(a_1, \dots, a_m) = f(\pi(a_1, \dots, a_n)) = 0$$

$$\therefore f \in I(\pi(W))$$

$$\Rightarrow \overline{\pi(W)} \subseteq V(J)$$

Now show $V(J) \subseteq \overline{\pi(W)}$

Suppose $f \in I(\pi(W)) \subseteq K[x_1, \dots, x_m]$

$$\Rightarrow f(a_1, \dots, a_m) = 0 \quad \forall (a_1, \dots, a_m) \in \pi(W)$$

Now consider f as a polynomial in $K[x_1, \dots, x_n]$

then certainly $\underbrace{f(a_1, \dots, a_m, a_{m+1}, \dots, a_n)}_{\text{since } f \text{ only depends on } x_1, \dots x_m} = 0 \quad \forall (a_1, \dots, a_n) \in W$

$\therefore f \in I(W) = \sqrt{I}$

↑ By the Nullstellensatz

and we assumed I was radical $\therefore I(W) = I$

and $f \in K[x_1, \dots, x_m]$

$\therefore f \in J$

$\Rightarrow I(\pi(W)) \subseteq J$

$\Rightarrow V(I(\pi(W))) \supseteq V(J)$

$$\uparrow$$
$$\overline{\pi(W)}$$

$$\therefore \overline{\pi(W)} = V(J)$$

Elimination can be done via a Lex G.B.

Fix the lex monomial order on $k[x_1, .., x_n]$ $(m<n)$

with $x_1 < x_2 < \cdots < x_n$

- Compute reduced G.B for an ideal $I = (f_1, .., f_s)$
- select the polys in the G.B which contain only $x_1, .., x_m$

Thm | If G is a lexicographic Gröbner basis for an ideal I in $k[x_1, .., x_n]$ then its elimination ideal $(m<n)$

$$J = I \cap k[x_1, .., x_m]$$

has Gröbner basis $G' = G \cap k[x_1, .., x_m]$. If G is a reduced G.B. then G' is a reduced G.B. of J.