(Elimination Theorem)

Thm Let $I \subseteq k[x_1,\dots,x_n]$ be an ideal, $W = V(I)$ it's variety. (k-algebraically closed)
Let $m<n$, and π the projection $\pi: K^n \to K^m$, $\pi: (p_1,\dots,p_m,p_{m+1},\dots,p_n) \mapsto (p_1,\dots,p_m)$.

Then $\overline{\pi(W)} = V(J)$ is defined by the **elimination ideal**

$$J = I \cap K[x_1,\dots,x_m].$$

If I is radical or prime (or homogeneous) then so is J (I won't prove now)

Note that. this tells us that we can get coordinates for a projection

i.e if $W = V(I) \subseteq \mathbb{C}^2$ $\dim(I)=0$

$I \cap K[x] = \langle f(x)\rangle$

find all points in $\overline{\pi(W)}$

lift thes to W by subbing the x-solution into I

– what if $p \in \overline{\pi(W)}$ but $(p,*) \notin W$

Thm 3, sec 5, ch 4 of CLO

Thm (Extension theorem) Let $I = \langle f_1,\dots,f_s\rangle \subseteq K[x_1,\dots,x_n]$ be ideal, k alg. closed, and use the Lex monomial order $x_1 < \cdots < x_n$. Let $J = I \cap K[x_1,\dots,x_{n-1}]$. Also let $W = V(I)$ and $\pi: k^n \to k^{n-1}$. $(p_1,\dots,p_n) \mapsto (p_1,\dots,p_{n-1})$ write

$$f_i = c_i(x_1,\dots,x_{n-1})x_n^{N_i} + \text{lower terms}$$

where $N_i \geq 0$ and $c_i \in k[x_1,\dots,x_{n-1}]$ $c_i \neq 0$

suppose $(a_1,\dots,a_{n-1}) \in V(J) = \overline{\pi(W)}$

If $(a_1, \dots, a_{n-1}) \in V(c_1, \dots, c_g)$ then $\exists\ a_n \in K$ s.t. $(a_1, \dots, a_n) \in V(I)$.

Proof c/c

In particular if we have a zero dim ideal and $\{g_1, \dots, g_n\}$ is its G.B. in Lex $x_1 < \dots < x_n$

and $g_i \in k[x_1, \dots, x_i]$

$LT(g_i) = c_i\, x_i^{N_i}$, $c_i \in k$ we can always find point in $V(I)$ by back subsitution.

Finding Numerical solutions to Poly. eqs via Homotopy continuation

work over $\mathbb{C}$

1-variable

Consider $f(x) = x^d + a_1 x^{d-1} + \dots + a_d$

Idea | start with the solutions to some other degree d polynomial, $g(x) = 0$, and track a path from this solutions to those $f(x) = 0$.

- we know that $g(x) = x^d - 1$ has solutions

$$x_k = e^{\frac{2\pi k i}{d}} \quad \text{for } k = 1, \dots, d$$

So lets define a homotopy

$$H(x,t) = t\, \underbrace{g(x)}_{\text{start system}} + (1-t)\, \underbrace{f(x)}_{\text{target system}}$$

So $H(x,1) = g(x)$, $H(x,0) = f(x)$

Idea: Numerically track (approximate) the solution paths from a solution to $H(x,1)=0$ to a solution of $H(x,0)=0$.

Let $\hat{x}_j(t)$ denote such a solution path, i.e. $\hat{x}_j(t)$ is a function of t s.t.

$$H(\hat{x}_j(t),t)=0 \quad \forall t\in[0,1]$$

when $t=1$, $\hat{x}_j(1)=e^{\frac{2\pi j i}{d}}$, when $t=0$ $f(\hat{x}_j(0))=0$

$\hat{x}_j(0)$ is a root of f.

Since we have

$$H(\hat{x}_j(t),t)=0 \quad \forall t\in[0,1]$$

Let $H_x(x,t)$ and $H_t(x,t)$ denote partial derivatives

$$0=\frac{dH(\hat{x}_j(t),t)}{dt}=H_x(\hat{x}_j(t),t)\frac{d\hat{x}(t)}{dt}+H_t(\hat{x}_j(t),t)$$

$$\frac{d\hat{x}_j(t)}{dt}=\frac{-H_t(\hat{x}_j(t),t)}{H_x(\hat{x}_j(t),t)}$$

$$\boxed{\frac{d\hat{x}_j(t)}{dt}=-\frac{g(\hat{x}_j(t))+f(\hat{x}_j(t))}{t\,g'(\hat{x}_j(t))+(1-t)f'(\hat{x}_j(t))}}$$

Find a numerical solution of this ODE for $t\in[0,1]$

we could use any ODE solver to get $\hat{x}_j(t)$.

we want to use that $H(\hat{x}_j(t),t)=0 \quad \forall t$.

Now work in $\mathbb{C}[x_1, \ldots, x_n]$ and consider a square system

$\uparrow$ # vars = # eqs

$f_1 = \cdots = f_n = 0$, i.e. $V(f_1, \ldots, f_n)$

and suppose $\langle f_1, \ldots, f_n \rangle$ is zero dimensional and radical and let $d_i = \deg(f_i)$

Then Define a Homotopy

$$H(x,t) = \begin{bmatrix} H_1(x,t) \\ \vdots \\ H_n(x,t) \end{bmatrix} = t \underbrace{\begin{bmatrix} x_1^{d_1} - 1 \\ \vdots \\ x_n^{d_n} - 1 \end{bmatrix}}_{g(x)} + (1-t) \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}}_{f(x)}$$

$g(x)$ ~ Start system

$f(x)$ ~ target system

Note that $\deg(V(g_1, \ldots, g_n)) = d_1 \cdots d_n$

Again we know the solutions to $H(x,1) = g(x) = 0$ (they are roots of unity)

$H(x,0) = f(x) = 0$ is the system we want to solve

as before

$$\tilde{x}^{(j)}(t) = \begin{bmatrix} \tilde{x}_1^{(j)}(t) \\ \vdots \\ \tilde{x}_n^{(j)}(t) \end{bmatrix} \quad \text{denote}$$

the solution path from the j^{th} point in $V(g_1, \ldots, g_n)$

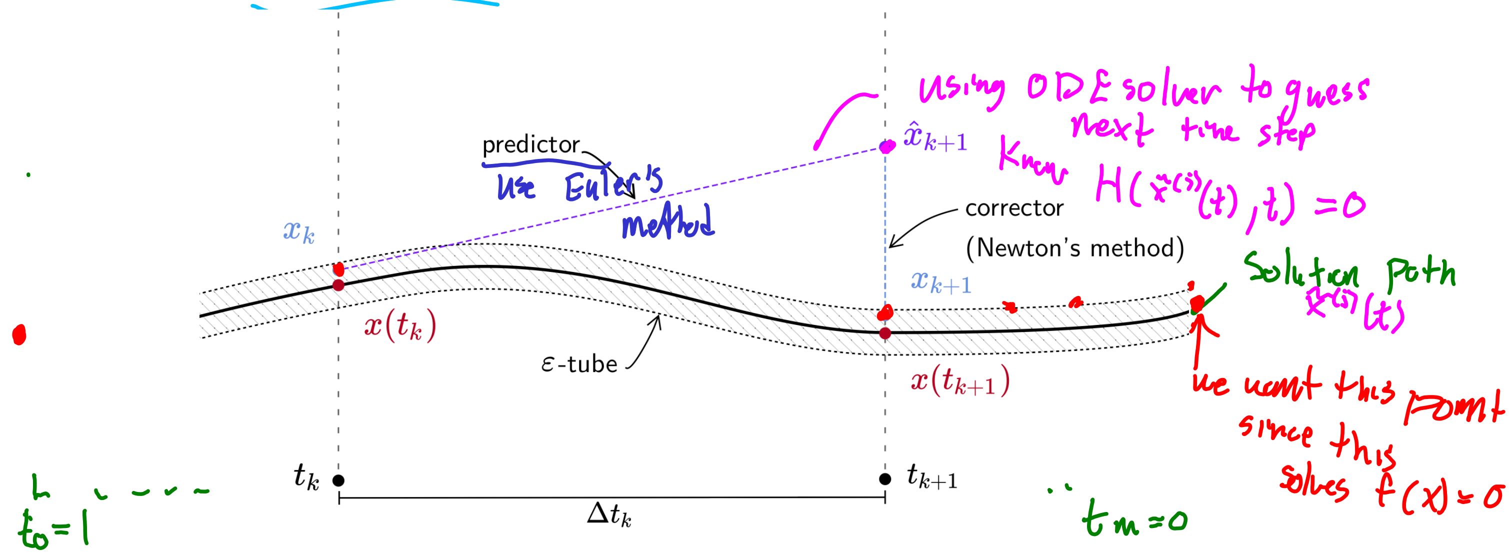
$$H(\tilde{x}^{(j)}(t), t) = 0 \quad \forall t \in [0,1] \quad \text{by def.}$$

Differentiating

$$\frac{d\tilde{x}^{(j)}(t)}{dt} = \begin{bmatrix} \frac{d\tilde{x}_1^{(j)}(t)}{dt} \\ \vdots \\ \frac{d\tilde{x}_n^{(j)}(t)}{dt} \end{bmatrix} = \left[t\,\mathrm{Jac}_x\left(g\left(\tilde{x}^{(j)}(t)\right)\right) + (1-t)\,\mathrm{Jac}_x\left(f\left(\tilde{x}^{(j)}(t)\right)\right) \right]^{-1} \begin{bmatrix} f_1(\tilde{x}^{(j)}(t)) - g_1(\tilde{x}^{(j)}(t)) \\ \vdots \\ f_n(\tilde{x}(t)) - g_n(\tilde{x}^{(j)}(t)) \end{bmatrix}$$

$\therefore$ our solution path $\tilde{x}^{(j)}(t)$ is the solution to this ODE in $t \in [0,1]$.

Our method of choice to solve this ODE is called <u>continuation</u>



we discretize $[0,1]$ into m peices

$$t_0, \ldots, t_m \quad , \quad h = \frac{1}{m}$$

$$t_0 = 1, \; t_m = 0 \quad , \quad t_j = (m-j)h$$

<u>Euler's method</u>

if we want to solve

$$\frac{d\hat{x}(t)}{dt} = W(\tilde{x}(t), t)$$

Eulers method, starting $t_0 = 1$ with step size

$$h = \frac{1}{m}, \quad t_j = (m-j)h$$

is

$$\tilde{X}(t_{j+1}) \simeq w_{j+1} = w_j + h\,W(w_j, t_j)$$

our initial condition $\tilde{x}(t_0) = \tilde{x}(1) =$ a root of unity $g(x) = x^d - 1$

in n-variables

$w^{(j)}$ is approx of $\tilde{x}(t_j)$

$$\tilde{X}(t_{j+1}) \simeq \begin{bmatrix} w_1^{(j+1)} \\ \vdots \\ \vdots \\ w_n^{(j+1)} \end{bmatrix} = \begin{bmatrix} w_1^{(j)} \\ \vdots \\ w_n^{(j)} \end{bmatrix} + h \begin{bmatrix} W_1(w^{(j)}, t_j) \\ \vdots \\ W_n(w^{(j)}, t_j) \end{bmatrix}$$

Eulers method can be derived from a first order Taylor expansion

$$\tilde{X}(t_0 + h) = \tilde{x}(t_0) + h \underbrace{\frac{d\tilde{x}}{dt}(t_0)}_{= W(\tilde{x}(t_0), t_0)} + O(h^2)$$

we will suplement this with a Newton's method corrector

Supplement with Newtons method corrector

i.e. instead

approx for $\tilde{X}(t_j)$

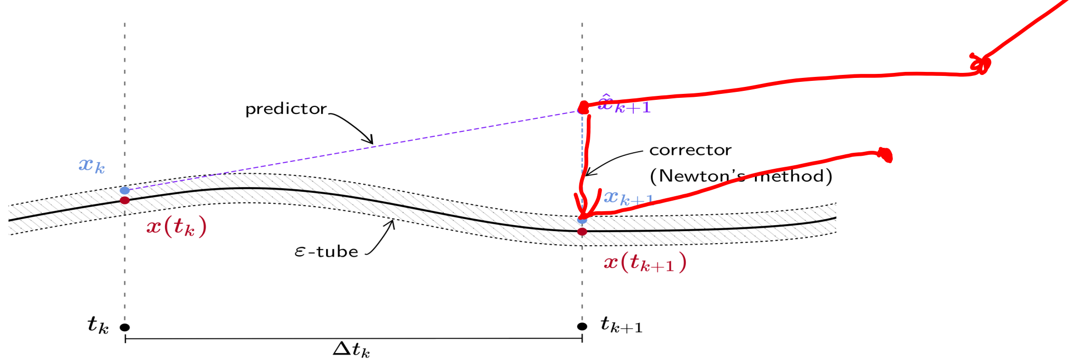
$$w_{j+1} = w_j + h\,W(w_j, t_j)$$

instead of using the last w_j from euler's method, correct it

use x^* in place of w_j

Compute x^* such that $H(x^*, t_j) = 0$

Using Newton's with start value w_j



Newton's method | Lets us find numerical aproximations to a system $\begin{bmatrix} w_1(x^*) \\ w_n(x^*) \end{bmatrix} = 0$ given an initial guess v

find x^*

$$w_{l+1} = w_l - \left[Jac_x\left(H(w_l, t_j) \right) \right]^{-1} H(w_l, t_j)$$

Starting at $w_0 = v \in \mathbb{C}^n$

$$w_{l+1} = w_l - \frac{t_j\, g(w_l) + (1-t_j) f(w_l)}{t_j\, g'(w_l) + (1-t_j) f'(w_l)}$$

This will converge to a solution $x^* \approx w_l$ for some l

when $x^* \approx \tilde{x}(t_j)$

Simple path tracker (ugly)

Set up | $f(x)$ = a degree d polynomial

m = # of time steps

$g(x) = x^d - 1$, $h = \frac{1}{m}$, $t_0 = 1$, $t_m = 0$

$t_\ell = (m-\ell)h$

Homotopy | $H(x,t) = e^{i\theta} t\, g(x) + (1-t) f(x)$, for a fixed $\theta \in (0, 2\pi)$

Output | Numerical approximations $\{w_0^{(1)}, \dots, w_m^{(1)}\}, \dots, \{w_0^{(d)}, \dots, w_m^{(d)}\}$

of the solution paths $\tilde{x}^{(1)}(t), \dots, \tilde{x}^{(d)}(t)$

such that $w_\ell^{(j)} \simeq \tilde{x}^{(j)}(t_\ell)$ and $\tilde{x}^{(j)}(1) = e^{\frac{2\pi j i}{d}}$.

Alg |

For j from 1 to d do:

- set $w_0^{(j)} := \hat{x}^{(j)}(1) = e^{\frac{2\pi j i}{d}}$
- For ℓ from 0 to $m-1$ do

← Predictor

(i) [one step of Euler's method]

$$w = w_\ell^{(j)} + h\left(\frac{-g(w_\ell^{(j)}) + f(w_\ell^{(j)})}{t\, g'(w_\ell^{(j)}) + (1-t) f'(w_\ell^{(j)})}\right)$$

(ii) [One run of Newton's method starting at w to find $w_{\ell+1}$]

"Solve" $H(x^*, t_{j+1}) = 0$

Iterate $a = 0, \dots$ while tol erance not achieved with $V_0 = w$

$$V_{a+1} = V_a - \frac{t_{j+1}\, g(v_a) + (1-t_{j+1}) f(v_a)}{t_{j+1}\, g'(v_a) + (1-t_{j+1}) f'(v_a)}$$

Stop when $|H(v_{a+1}, t_{j+1})| < tol$

Set $u^{(i)}_{d+1} = v_{d+1}$.

in $\mathbb{C}[x_1,\dots,x_n]$ we use multi-variable Newton / Euler method.

problems include

- how do we choose step size $h = \frac{1}{m}$
- what if Newton's method fails?
- what about multiple roots?

$\hookrightarrow x^2=0$

"Gamma trick"

$\gamma = e^{i\theta}$

Having $\widetilde{e^{i\theta}}$ in $H(x,t) = t\,\widetilde{e^{i\theta}}\,g(x) + (1-t)f(x)$

lets us solve systems like $x^2+1=0$.

Theorem 8.4.1 of S.W.

Thm work in $\mathbb{C}[x_1,\dots,x_n]$. Let $I = \langle f_1,\dots,f_n\rangle$ be an ideal s.t. $\dim(I)=0$. Set $d_i = \deg(f_i)$

Let $g_1 = x_1^{d_1}-1, \dots, g_n = x_n^{d_n}-1$, $d = d_1 d_2 \cdots d_n$

(Note $\deg(V(f_1,\dots,f_n)) \leq d$)

Then the d solution paths $\hat{x}^{(1)}(t),\dots,\hat{x}^{(d)}(t)$ of the homotopy

$$H(x,t) = \gamma t\, g(x) + (1-t) f(x) = 0$$

$$= \gamma t \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} + (1-t) \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = 0$$

starting at points in $V(g_1,\dots,g_n)$ are non-singular for $t \in (0,1]$

and their endpoints as $t \to 0$ include all points in $V(f_1, \dots, f_n)$ for almost all $\gamma \in \mathbb{C}$ except for a finite number of real 1-dim rays through the origin in complex plane. In particular if $\gamma = e^{i\theta}$ then all but finitely many θ work.