Def Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal, its real radical

$$\sqrt[\mathbb{R}]{I} = \left\{ f \in \mathbb{R}[x_1,\dots,x_n] \;\middle|\; f^{2m} + g_1^2 + \dots + g_s^2 \in I \text{ for some } m \in \mathbb{N} \text{ and some } g_1, \dots, g_s \in \mathbb{R}[x_1,\dots,x_n] \right\}$$

Thm (Real Nullstellensatz) For any ideal $I = \langle f_1, \dots, f_r \rangle$ in $\mathbb{R}[x_1, \dots, x_n]$ we have

$$I(V_{\mathbb{R}}(I)) = \sqrt[\mathbb{R}]{I}$$

Proof First show that $\sqrt[\mathbb{R}]{I} \subseteq I(V_{\mathbb{R}}(I))$.

If $f(x) \in \sqrt[\mathbb{R}]{I} \Rightarrow \underbrace{f^{2m} + g_1^2 + \dots + g_s^2}_{} \in I$

$\therefore$ this must vanish on all points in $V_{\mathbb{R}}(I)$

Let $a \in V_{\mathbb{R}}(I)$

then $\underbrace{(f(a))^{2m}}_{\geq 0} + \underbrace{g_1(a)^2 + \dots + g_s(a)^2}_{\geq 0} = 0$

$\Rightarrow (f(a))^{2m} = 0$ and $g_i(a) = 0 \;\; \forall \; a \in V_{\mathbb{R}}(I)$

$\Rightarrow f(a) = 0 \Rightarrow f \in I(V_{\mathbb{R}}(I))$.

Now suppose that $f(x) \in I(V_{\mathbb{R}}(I))$. Consider

$$J = \langle \underbrace{f_1, \dots, f_r}_{\text{gens of } I}, yf - 1 \rangle \subseteq \mathbb{R}[x, y]$$

Note $V_{\mathbb{R}}(J) = \phi$ (in fact $V_{\mathbb{C}}(J) = \phi$)

$\therefore$ By the real weak Nullstellensatz

$$1 + p_1^2 + \cdots + p_r^2 \in J \quad \text{for some } p_r \in \mathbb{R}[x_1, \ldots, x_n, y]$$

Again sub in $y = \frac{1}{f}$ and clear denominators

$$\left(p_i\left(x, \tfrac{1}{f}\right)\right)^2 = \left(\frac{q_i(x)}{f(x)^{m_i}}\right)^2 \qquad m_i = \deg_y p_i$$

$\therefore$ for some m

$$\Rightarrow \quad (f(x))^{2m} + \tilde{p}_1(x)^2 + \cdots + \tilde{p}_r(x)^2 \in J \cap \mathbb{R}[x_1, \ldots, x_n]$$

and $I = J \cap \mathbb{R}[x_1, \ldots, x_n]$

$$\therefore \quad f^{2m} + \tilde{p}_1^2 + \cdots + \tilde{p}_n^2 \in I$$

$$\therefore \quad f \in \sqrt[\mathbb{R}]{I} \quad . \blacksquare$$

Toric Varieties

- A toric variety is an irreducible variety that is parametrized by a vector of monomials i.e. $\exists$ a full rank $d \times n$ integer matrix A s.t. $A = [a_1 \cdots a_n]$ ← columns

$$X_A = \overline{\left\{ (t^{a_1}, \ldots, t^{a_n}) \in K^n \;\middle|\; t \in (K^*)^d \right\}}$$

$t^{a_1} = t_1^{a_{11}} \cdots t_d^{a_{d1}}$

$K^* =$ non-zero elements of K

- An irreducible variety X is toric iff $I(X)$ = prime ideal generated by binomials, $x^b - x^c$ for $b, c \in \mathbb{N}^n$

$x_1^{b_1} \cdots x_n^{b_n}$

The monomials (and binomials) correspond to points on an integer lattice (i.e. in $\mathbb{Z}^d$) which allows us to compute many things about toric varieties just from

$$P = \mathrm{conv}(A)$$

Euclidean volume

$$\deg(X_A) = (\dim(P)!) \cdot \mathrm{Vol}(P)$$

$$\dim(X_A) = \dim(P)$$

Why Toric?

This comes from the algebraic torus. Let k be an algebraically closed field, and consider the Laurent Polynomial Ring $k[x_1^{\pm}, \ldots, x_n^{\pm}]$

The associated variety

$$(k^*)^n = \mathrm{Spec}\, k[x_1^{\pm}, \ldots, x_n^{\pm}]$$

Algebraic Torus of dimension n, over k.

In the special case $n=2$, $k=\mathbb{C}$

$$(\mathbb{C}^*)^2 \cong (\underbrace{\mathbb{R}_+ \times S^1}_{S^1 = \text{unit circle}})^2$$

Topological torus $S^1 \times S^1$ = alg torus mult by the contractable factor $\mathbb{R}_+ \times \mathbb{R}_+$

$(k^*)^n$ is a group under coordinate wise multiplication

Def 1 (Character of a torus) A <u>character</u> of the alg. torus $T=(k^*)^n$ is an algebraic map $\chi : T \to k^*$ which is also a group homomorphism.

Lemma 1 Every character χ of the torus $T=(k^*)^n$ is given by a Laurent monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ for some $b \in \mathbb{Z}^n$

$\therefore$ characters of T can be thought of as elements of $\mathbb{Z}^n$.

$T = (k^*)^n$ = alg. torus

Characters of T will be equiv. $x^b = x_1^{b_1} \cdots x_n^{b_n}$, $b \in \mathbb{Z}^n$

or $b \in \mathbb{Z}^n$

Under this correspondence multiplication of characters (i.e. monomials) becomes in the group $(\mathbb{Z}^n, +)$

$$\chi_1(x) \cdot \chi_2(x) = (\chi_1 + \chi_n)(x)$$

A group isomorphic to $(\mathbb{Z}^l, +)$ is called a lattice

The lattice of characters of T will be denoted M_T or M

Since a subgroup of a free abelian group is free

$\therefore$ any set of characters generates a sublattice $\hat{M} \subseteq M \cong \mathbb{Z}^n$

subgroup $\cong \mathbb{Z}^l$, $l \leq n$

Let $a_1, \dots, a_p$ be characters in $M_T \cong \mathbb{Z}^n$

Let $A = [a_1 \ \dots \ a_p] = n \times p$ matrix whose columns are the vectors a_i.

$\hat{M}$ = sublattice generated by $a_1, \dots, a_p$

= image of $\mathbb{Z}^p$ under mult by A $(A : \mathbb{Z}^p \to \mathbb{Z}^n)$

Prop The image of $T = (k^*)^n$ in $(k^*)^p$ under the monomial map $f : x \mapsto x^A \cong (x^{a_1}, \dots, x^{a_p})$

$x_1^{a_{11}} \cdots x_n^{a_{n1}}$

is also a torus $\tilde{T}$. The character lattice of $\tilde{T}$ is $\tilde{M}$

<u>Proof</u> | The monomial map $f : T_x \longrightarrow (k^*)^p_y$

induces a ring homomorphism

$$f^* : k[y_1^{\pm}, \dots, y_p^{\pm}] \longrightarrow k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$y_i \longmapsto x^{a_i}$$

$\mathrm{im}(f^*) = k[x^{a_1}, \dots, x^{a_p}] \cong k[\tilde{M}]$ group algebra generated by $\tilde{M}$

$\mathrm{Spec}(\mathrm{im}(f^*)] \subseteq (k^*)^p$

$\tilde{M} \cong \mathbb{Z}^d$ for some $d \leq \min(n, p)$
$\quad \llcorner = \mathrm{Rank}(A)$

Note that $\mathrm{Spec}(\mathrm{im}(f^*)] = \tilde{T}$, by def

So $\tilde{T} = \mathrm{Spec}(k[\tilde{M}]) \subseteq (k^*)^p$
$\qquad \qquad \| $
$\qquad (k^*)^d$ since $\tilde{M} \cong \mathbb{Z}^d$. ∎

we are interested in the Zariski closure in k^p of the d-dim torus $\tilde{T}$ above

<u>Def</u> (Affine toric variety) An affine toric variety is the Zariski closure in k^p of the image of f above, ie

$$X_A := \overline{\{(x^{a_1}, \dots, x^{a_p}) \mid x \in (k^*)^n\}} \subseteq k^p$$

for A = an $n \times p$ integer matrix.

The p columns of A represent characters of $T = (k^*)^n$

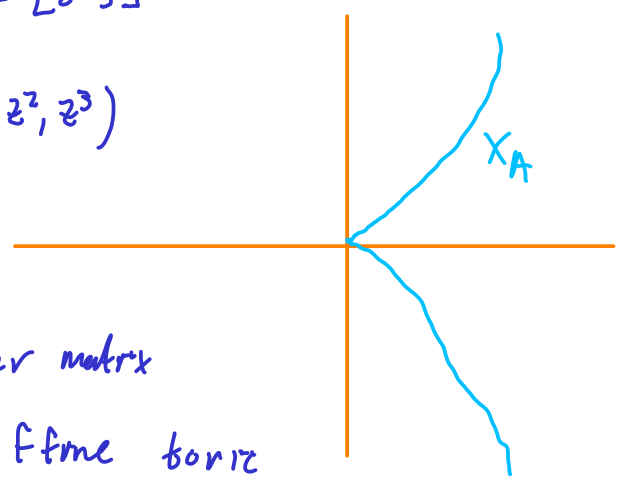
Ex] (Affine space is toric)

$$k^p = X_{\underset{\llcorner \text{ the } p\times p \text{ identity matrix}}{id}}$$

Ex] $V(x^3 - y^2)$ is a toric variety

$= X_A$ for $A = [2 \ 3]$

monomial map is $z \longmapsto (z^2, z^3)$



Let $A = [a_1, \ldots, a_p]$ by a $n \times p$ interger matrix

Prop] The dimension of the affine toric variety X_A is

$$\dim(X_A) = \operatorname{rank}(\hat{M}) \quad \text{lattice spanned by } a_1, \ldots, a_p$$
$$= \operatorname{rank}(A).$$

Proof] In the last prop. we showed that

$\hat{T}$ = image of T in $(k^*)^n$
$= (k^*)^d$

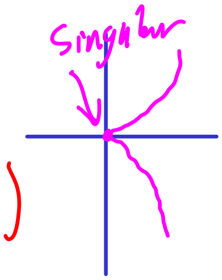
, $d = \operatorname{Rank}(\hat{M}) = \operatorname{Rank}(A)$.

and $X_A = \overline{\hat{T}}$ by definition

$\therefore \quad \dim(X_A) = \dim(\tilde{T}) = \dim(T) = d.$

Def) Given a set of characters $\{a_1, \dots, a_p\}$ of the alg. torus $T = (k^*)^n$, the <u>monoid S</u> generated by $\{a_1, \dots, a_p\}$ in $M_T \cong \mathbb{Z}^n$ is the smallest set containing 0, $\{a_1, \dots, a_p\}$ and which is closed under addition.

In Prop 1, we showed $X_A = \operatorname{Spec} k[S]$



Ex) The cuspidal cubic $X_A = V(x^3 - y^2)$

$A = [2, 3]$ $\quad \therefore$ The ass. monoid $S = \{0, 2, 3, 4, 5, \dots\}$

a missing bit.

Ex) k^1 = affine line $= X_A$, $A = [1]$

and the ass. monoid is $S = \{0, 1, 2, 3, \dots\}$

Def) A submonoid S in a lattice M is called <u>saturated</u> iff for any $x \in M$, $k \in \mathbb{Z}_+$ we have

$$kx \in S \Rightarrow x \in S$$

An affine toric variety $X_A = \operatorname{Spec}(k[S])$ for which S is saturated in $\tilde{M}$ = lattice gen. by $a_1, \dots, a_p$ is called <u>normal</u>.

Varieties that are <u>not</u> normal are always singular.

Note also that $\underbrace{(k^*)^d}_{\dim = d} \subseteq X_A \subseteq k^p$

$(k^*)^d = X_A - V(y_1 \cdots y_p)$

this is smooth

$\Rightarrow$ any singular point of X_A has a zero coordinate.

The prime ideal of a toric variety

$X_A \subset k_y^p$, $f : x \mapsto x^A$, so $X_A = \overline{f(T)}$

$$f^* : k[y_1^{\pm 1}, \ldots, y_p^{\pm 1}] \longrightarrow k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

we will see that $I(X_A) = \ker(f^*)\big|_{k[y_1, \ldots, y_p]}$

Lemma Let $A = [a_1, \ldots, a_p]$, X_A the associated toric variety. Then

1) Any relation $\sum_i b_i a_i = \sum_j c_j a_j$, $b_i, c_j \in \mathbb{Z}_+$ gives a binomial $y_1^{b_1} \cdots y_p^{b_p} - y_1^{c_1} \cdots y_p^{c_p} \in I(X_A)$

2) Every binomial in $I(X_A)$ has this form.

3) $I(X_A)$ is generated by these binomials.

Proof sketch A polynomial in $k[y_1, \ldots, y_p]$ vanishes at every point in a toric variety iff we obtain the zero polynomial after subbing y_i by x^{a_i}.

For binomials in particular, doing the sub $y_i = x^{a_i}$ turns monomials in y into monomials in x, and a binomial in x's will be the zero poly iff the exponents of the two monomials satisfy the relation

$$\sum b_i a_i = \sum c_j a_j$$

$\therefore$ all Binomials are as in (1).

3) is proved by induction on the number of terms with the base case being the binomial case.

All together Every toric variety is

$X_A = V(I_{X_A})$ for I_{X_A} a binomial ideal generated by $y^b - y^c$ for $(b-c) \in \ker(A)$

Thm) Let $I = \langle y^{b_i} - y^{c_i} \mid b_i, c_i \in \mathbb{Z}^p_{\geq 0}, i \in I \subseteq \mathbb{N} \rangle$ be a prime ideal. Then $V(I) = X_A$ for some matrix A.

Proof) By H.B.T we know $\exists$

a finite subset of these generators which generates I

$$\therefore \quad I = \langle y^{b_1} - y^{c_1}, \dots, y^{b_s} - y^{c_s} \rangle$$

Since I is prime the integer vectors b_i, c_i must have disjoint support (otherwise $y^{b_i} - y^{c_i}$ would not be irreducible)

Let $B = [b_1 - c_1, \dots, b_s - c_s]$ this is a $p \times s$ matrix

Let A be a $n \times p$ matrix s.t.

$\text{Row}(A) = \ker(B^T)$ (where $\ker(B^T)$ is understood as a $\mathbb{Z}$-module)

Claim $\text{col}(B) = \ker(A)$

This is true over $\mathbb{Q}$ or $\mathbb{R}$, but also holds over $\mathbb{Z}$ since I is prime

If not, $\Rightarrow$ $b - c \notin \text{col}(B)$ but $\ell b - \ell c \in \text{col}(B)$

take the smallest such ℓ then we have

$$\underbrace{y^{\ell b} - y^{\ell c}}_{\in I} = \underbrace{(y^b - y^c)}_{\substack{\notin I \\ \text{by assumption}}} \underbrace{\left(y^{(\ell-1)b} + y^{(\ell-2)b} y^c + \dots + y^{(\ell-1)c}\right)}_{\substack{\notin I \\ \text{since the coefficients} \\ \text{don't sum to zero (Char}(k) = 0)}}$$

This contradicts that I is a prime ideal!

$\therefore \quad X_A = V(I)$

$\uparrow$ A s.t. $Row(A) \supseteq \ker(B^T)$.

In Book affine toric varieties are related to combinatorial ob. called a cone (can also be related to fan's Cox, Little, Schenck, Toric varieties)

Let $A = [a_1, \dots, a_p]$ be a $n \times p$ integer matrix of rank n, with $(1, \dots, 1)$ in the row space of A.

$X_A = V(I_A)$

One can show, that the binomials are homogeneous since $(1,1,\dots,1)$ is in the row space of A.

$$\therefore \quad X_A = V(I_A) \subseteq P^{p-1}$$

ideal from prop $I_A = \langle y^b - y^c \mid b-c \in \ker(A) \rangle$

Projective toric variety

Def/ A <u>polytope</u> $P \subseteq \mathbb{R}^n$ is the convex hull of a finite set of points. It is a <u>lattice polytope</u> if it is the convex hull of points in $\mathbb{Z}^n$.

A <u>face</u> F of P is a subset of the form

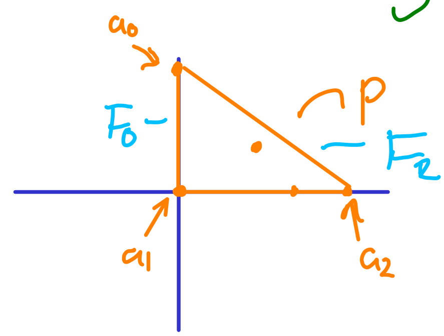
$$F = \{ p \in P \mid l(p) = 0 \}$$ where l is an affine linear poly which is non-negative on P

i.e, $\ell(c) \geq 0 \ \forall c \in P$.

The <u>Dimension</u> of a face F of a polytope P is the dimension of the smallest linear space containing F

$$A = \begin{array}{c} \begin{bmatrix} 0 & 0 & 3 & 2 & 2 \\ 4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccccc} a_0 & & & & \end{array} \end{array}$$

$P =$



A toric variety $X_A \subseteq \mathbb{P}^{p-1}$ is the closure in $\mathbb{P}^{p-1}$ of the torus $\tilde{T} \subseteq (k^\times)^p$

image of mon. map

$$X_A = \overline{\tilde{T}}$$

The group $\tilde{T}$ acts on both itself and on k^n, and on $X_A = \overline{\tilde{T}}$

<u>Def</u> | The <u>torus orbits</u> on X_A are the orbits of the action of $\tilde{T}$ on X_A.

<u>Thm</u> | For $X_A = V(I_A)$, a proj. toric var, $P = \operatorname{conv}(A)$

The torus orbits in X_A are in bijection with the faces of P. The orbit corresponding to a face F is

$$\mathcal{O}(F) := \{ y \in X_A \mid y_i \neq 0 \text{ iff } a_i \in F \}$$

The closure of this orbit is the proj toric variety

$$V(F) = \overline{\mathcal{O}(F)} = \{ [y_1 : \cdots : y_p] \mid y_i = x^{a_i} \text{ if } a_i \in F \text{ and } y_i = 0 \text{ if } a_i \notin F \}$$

$$\subseteq X_A$$

$$\dim(\mathcal{O}(F)) = \dim(V(F)) = \dim(F).$$

Thm | Let X_A be a projective toric variety, $P = \text{conv}(A)$

Then $\deg(X_A) = (\dim(P)!) \cdot \underbrace{\text{Vol}}_{\text{Euclidean vol}}(P)$ ($\mathbb{Z}A$ is an index one lattice in $\mathbb{Z}^n$)

Proved in Gröbner and Convex polytopes, by Sturmfels.