Assignment M2/Sage Question

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Work in three dimensional projective space $\mathbb{P}^3$ with coordinate ring $k[x, y, z, w]$. Consider the projective varieties

$$X = V(wx^2 - y^3) \subset \mathbb{P}^3, \ Y = V(y) \subset \mathbb{P}^3$$

and their intersection

$$W = X \cap Y = V(y, xw) = V(x, y) \cup V(y, w) \subset \mathbb{P}^3.$$

1. Compute $\deg(W) = \deg(I(W))$ where $I(W) = \langle y, xw \rangle$ is the (radical) ideal defining W.

2. Compute $\deg(X)$ and $\deg(Y)$.

3. Use M2/Sage to generate two random polynomials f, g with $\deg(f) = 3$, and $\deg(g) = 1$. Check that $\langle f, g \rangle$ is radical. Compute $\deg(V(f, g)) = \deg(V(f) \cap V(g))$. Explain what you observe with Bézout's Theorem.

4. Contrast the case of random polynomials with the case for $\deg(W) = \deg(X \cap Y)$ above. Can you explain what is happening here? To help explain the situation geometrically try the steps below:

   - Compute the singular locus $\text{Sing}(X)$ of X. How does that relate to the intersection $X \cap Y$ above?
   - Try to sketch what is happening, set $w = 1$ above and work in $\mathbb{R}^3$, what do you observe?

5. Use the M2 code below (in the M2 online calculator if you don't have M2 on your machine, I don't think Sage can do this).

```
needsPackage "SegreClasses"
R=QQ[x..z,w]
X=ideal(w*x^2-y^3)
multiplicity(ideal(x,y),X)
multiplicity(ideal(w,y),X)
```

   This assigns a multiplicity to each of the components of W. Let $W_1 = V(x, y)$ and $W_2 = V(y, w)$ so that $W = W_1 \cup W_2$ and let $e_{W_i}X$ be the multiplicity assigned to W_i inside X. Compare the number $e_{W_1}X \deg(W_1) + e_{W_2}X \deg(W_2)$ to the degree of $X \cap Y$ predicted by Bézout's theorem.