Assignment 1 Hints

Question 12 Hints

One can form a conjecture using the M2 commands (with different n values):

```
n=5
R=QQ[x_1..x_n]
I=ideal(for i from 1 to n list sum(subsets(gens(R),i),s->product(s)))
ideal groebnerBasis I
ideal leadTerm I
dim I
flatten entries basis (R/I)
```

For the example above you obtain $in_<(I) = \langle x_1, x_2^2, \dots, x_5^5 \rangle$.

The overarching strategy is to consider the set of standard monomials. Work in $R = k_<[x_1, \dots, x_n]$ over a field k with a monomial order $<$; `GrevLex` will work. Let e_j^n denote the elementary symmetric polynomial of degree j in n variables, i.e.

$$e_j^n = \sum_{r_1 \le r_2 \le \cdots \le r_j \le n} x_{r_1} x_{r_2} \cdots x_{r_j}.$$

Let $I_n = \langle e_1^n, \dots, e_n^n \rangle$. Consider the following proof strategy:

1. Show that the set of standard monomials is
$$S_<(I_n) = \{x_1^{a_1} \cdots x_n^{a_n} \mid a_j < j\}.$$
2. Use this to conclude that the initial ideal is as you conjectured above.

For the first step you may use the following fact without proof:

- The set $\{e_{i_1}^1 e_{i_2}^2 \cdots e_{i_{n-1}}^{n-1} \mid 0 \le i_\ell \le \ell\}$ is linearly independent in R/I_n; where $e_j^w \in k[x_1, \dots, x_n]$ (for $w = 0, \dots, n-1$) denotes the elementary symmetric polynomial in the variables $x_1, \dots, x_w$, e.g. $e_1^2 = x_1 + x_2$

Using this you can show $\{e_{i_1}^1 e_{i_2}^2 \cdots e_{i_{n-1}}^{n-1} \mid 0 \le i_\ell \le \ell\}$ is a basis for R/I_n, and then show that the set of standard monomials of I_n is as above. In this part of your proof you may find it helpful to use at least one (not necessarily all) of the following identities (you may also use these without proof, most are pretty easy to check, the last you would need to do some induction):

- $e_\ell^i = e_\ell^{i-1} + x_i e_{\ell-1}^{i-1}$
- $(e_i^{\ell+1} - e_i^\ell) e_{j-1}^\ell = (e_j^{\ell+1} - e_j^\ell) e_{i-1}^\ell$
- $x_j = e_1^j - e_1^{j-1}$.
- for any i, j, ℓ we have
$$e_i^\ell e_j^\ell = \sum_{\nu \ge 0} e_{i-\nu}^{\ell+1} e_{j+\nu}^\ell - \sum_{\nu \ge 1} e_{i-\nu}^\ell e_{j+\nu}^{\ell+1}.$$

Question 13

You don't need to give a proof, but do give a justification for what you observe in terms of the Hilbert polynomial and/or Hilbert function and/or Hilbert series.