Elimination can be done via a Lex G.B.

Fix the lex monomial order on $K[x_1,\dots,x_n]$ $(m<n)$

with $\underline{x_1 < x_2 < \cdots < x_n}$

- Compute reduced GB for an ideal $I = (f_1,\dots,f_s)$
- select the polys in the GB which contain only $x_1,\dots,x_m$

Thm | If G is a lexicographic Gröbner basis for an ideal I in $K[x_1,\dots,x_n]$ then its elimination ideal $(m<n)$ corresponds to $\pi: K^n \to K^m$, $m<n$

$$J = I \cap K[x_1,\dots,x_m]$$

$\overline{\pi(V(I))} = V(J)$

has Gröbner basis $G' = G \cap K[x_1,\dots,x_m]$. If G is a reduced G.B. then G' is a $\underline{\text{reduced G.B.}}$ of J.

Proof |

Clearly $G' \subseteq J = I \cap K[x_1,\dots,x_m]$ since $G \subseteq I$

Now show that G' is a Gröbner basis for J

That is, show that some initial monomial from G' divides $in_<(f)$ $\forall f \in J - \{0\}$.

Let $f \in J$, $f \neq 0$. $in_<(f)$ is divisible by $in_<(g)$ for some $g \in G$. None of the monomials $x_{m+1},\dots,x_n$ can appear in $in_<(g)$ since $x_{m+1},\dots,x_n$ are larger than any monomial in only $x_1,\dots,x_m$, but $in_<(f)$ only contains $x_1,\dots,x_m$

$\therefore$ $in_<(g) \in K[x_1,\dots,x_m]$ and $x_{m+1},\dots,x_n$ are larger than $\underline{\text{any}}$ monomial in $K[x_1,\dots,x_m]$

$\therefore$ all monomials in g are in $K[x_1,\dots,x_m]$

$\therefore$ $g \in K[x_1,\dots,x_m]$ and $g \in G'$

$\therefore \forall f \in J - \{0\}$

$\exists g \in G'$ s.t. $in_<(g)$ divides $in_<(f)$

$\therefore$ G' is a Gröbner basis of J.

Since all polys in G' arise directly from G then if G is reduced then so is G'. ∎

- Note that since the G.B of a homogeneous ideal is homogeneous then we can do projections $\pi : \mathbb{P}^n \to \mathbb{P}^m$ in the same way, i.e. $\overline{\pi(V(I))} = V(J)$

$J = I \cap k[x_0, \dots, x_m]$