Last week we looked at

How to find all points in

$V(I) \subseteq \mathbb{C}^n$ provided $\dim(I) = 0$.

Numerical Irreducible Decomposition

$I = \langle f_1, \dots, f_w \rangle$

Suppose $X = V(I) \subseteq \mathbb{C}^n$, $\dim(X) = m < n$

Suppose I is radical

with $X = V_1 \cup \cdots \cup V_r$ where V_i is irreducible

Suppose $\dim(V_r) = \dim(X)$ and $\dim(V_1) < \dim(V_2) < \cdots < \dim(V_r)$

Let $H_i \subseteq \mathbb{C}^n$ be a general hyperplane, $H_i = V(\ell_i)$, ℓ_i ~ general linear poly.

Cardinality

Then $\deg(X) = \# (X \cap H_1 \cap \cdots \cap H_m)$

$= \# (V_r \cap H_1 \cap \cdots \cap H_m)$

since $V_i \cap H_1 \cap \cdots \cap H_m = \emptyset$ for $i < r$

Can be computed via homotopy continuation

by solving the system $f_1 = \cdots = f_w = \ell_1 = \cdots = \ell_m = 0$

the # of solutions to it's $\deg(X)$.

Further Note

$\dim(X) = \dim(V_r)$ is the $\max j \in \{1, \dots, n\}$

s.t. $X \cap H_1 \cap \cdots \cap H_j$ is non-empty

and if $X \cap H_1 \cap \cdots \cap H_{j+1} = \emptyset$

then applying homotopy continuation to

$f_1 = \cdots = f_w = l_1 = \cdots = l_{j+1}$ will give no solutions

$\therefore$ $\deg(X)$ and $\dim(X)$ can be computed via homotopy continuation.

Now consider V_{r-1}, $\dim(V_{r-1}) < m$, say $\dim(V_{r-1}) = m-1$

Then apply homotopy continuation to

$$f_1 = \cdots = f_w = l_1 = \cdots = l_{m-1}$$

will give all points in V_{r-1} (in fact $\deg(V_{r-1})$ of them) plus some points in V_r

To see this Do a prime decomp of I

the prime component $\langle g_1, \dots, g_v \rangle = P_{r-1}$ $(V(P_{r-1}) = V_{r-1})$ should give $\deg(V_{r-1})$ Solutions to

$$g_1 = \cdots = g_v = l_1 = \cdots = l_{m-1}$$

and those are among the points in $I + \langle l_1, \dots, l_{m-1} \rangle$

Now we have $\deg(V_{r-1})$ points + other points

- There is a test called the Trace test which allows us to check if a set of points are in the same irreducible component.

Using this we pick out the $\deg(V_{r-1})$ points from V_{r-1} in the solutions $f_1 = \cdots = f_w = l_1 = \cdots = l_{m-1}$.