Def If q is a primary ideal in a Noetherian ring R, $p = \sqrt{q}$ we say q is p-primary.

Def A minimal primary decomposition is

$$I = q_1 \cap q_2 \cap \cdots \cap q_l$$

s.t.

- q_i is primary
- $\sqrt{q_i} \neq \sqrt{q_j}$ for $i \neq j$ these are unique
- $\bigcap_{j \neq i_0} q_j \nsubseteq q_{i_0}$ $\forall$ $1 \le i_0 \le l$.

Note $\langle x^2, xy \rangle = \langle x \rangle \cap \langle x, y \rangle^2 = \langle x \rangle \cap \langle x^2, y \rangle$ is an example of 2 minimal primary decompositions.

Lemma 1 If p is a prime ideal in a ring R, $q_1, \ldots, q_r$ are ideals and

$$p \supseteq q_1 \cap \cdots \cap q_r \Rightarrow p \supseteq q_j \text{ for some } j \in \{1, \ldots, r\}$$

and further if q_j is prime $\forall j \Rightarrow p = q_j$ for some j.

Lemma 2 In a Noetherian ring every ideal contains a power of its radical.

Def

Recall $I : a = \{ b \in R \mid ab \in I \}$ ($a \in R$, I an ideal) colon ideal or ideal quotient

Thm For any ideal in a ring R with minimal primary decomposition

$$I = q_1 \cap \cdots \cap q_r$$

then the ideals $p_i = \sqrt{q_i}$ do not depend on the choice of decomposition. Further

$$\{p_1, \ldots, p_r\} = \{ p \subset R \mid p \text{ is a prime ideal and } p = \sqrt{I:a} \text{ for some } a \in R \}$$

If R is Noetherian

$$\{p_1, \ldots, p_r\} = \{ p \subset R \mid p = I:a \text{ is a prime ideal for some } a \in R \}.$$

Note if $a \in I$
$I:a = \{ b \in R \mid ab \in R \}$
$= R$

Proof of Thm

Outline of Proof:

- Show that if $\sqrt{I:a}$ is prime for some $a \in R$ then $\sqrt{I:a} = \sqrt{q_{i_0}}$ for some i_0
- Show for any q_j $\exists a \in R$ s.t. $\sqrt{I:a} = \sqrt{q_j}$

From this it follows that

$$\{\sqrt{q_1}, \ldots, \sqrt{q_r}\} = \{ \sqrt{I:a} \mid \sqrt{I:a} \text{ is prime and } a \in R \}$$

Note this set does not depend on the chosen decomposition.

Proof / Fix a decomposition $I = q_1 \cap \cdots \cap q_r$ then

$$I:a = \bigcap_{i=1}^{r} q_i : a = \bigcap_{\substack{i=1 \\ a \notin q_i}}^{r} q_i : a$$

Consider $\sqrt{I:a} = \bigcap_{\substack{j=1 \\ a \notin q_j}}^{r} \sqrt{q_j : a}$

Now show $a \notin q_j \Rightarrow \sqrt{q_j : a} = \sqrt{q_j}$

suppose $b \in \sqrt{q_j : a} \Rightarrow b^n \in q_j : a$
$\Rightarrow a b^n \in q_j$

Since q_j is primary & $a \notin q_j \Rightarrow (b^n)^m \in q_j$

$\Rightarrow b \in \sqrt{q_j}$

$\Rightarrow \sqrt{q_j : a} \subseteq \sqrt{q_j}$

and $\sqrt{q_j} \subseteq \sqrt{q_j : a}$ since $q_j \subseteq q_j : a$

$\therefore \sqrt{q_j} = \sqrt{q_j : a}$

each of these is prime

$\therefore \sqrt{I:a} = \bigcap_{a \notin q_j} \sqrt{q_j}$

Now suppose $\sqrt{I:a}$ is prime (since we are interested only in the set of $\sqrt{I:a}$ s.t. $\sqrt{I:a}$ is prime)

$\therefore$ Lemma 1 $\Rightarrow \sqrt{I:a} = \sqrt{q_i}$ for some i

Hence whenever $\sqrt{I:a}$ is prime it is equal to some $p_i = \sqrt{q_i}$.
Now we need only show all $p_1, \ldots, p_r$ arise in this way.

Consider some $\sqrt{q_{i_0}}$. Since the primary decomposition is minimal

$$\exists \quad a \in \left(\bigcap_{j \neq i_0} q_j \right) - q_{i_0}$$

Fix this a

then $\sqrt{I:a} = \bigcap_{a\notin q_j} \sqrt{q_j : a} = \bigcap_{a \notin q_j} \sqrt{q_j} = \sqrt{q_{i_0}}$

$\therefore$ every $\sqrt{q_{i_0}}$ arises as $\sqrt{I:a}$ for some a

$\therefore$ $\{ p \subset R \mid p \text{ is prime and } p = \sqrt{I:a} \text{ for } a \in R \}$ consists of only $\sqrt{q_1}, \dots, \sqrt{q_r}$

$\therefore$ $\{\sqrt{q_1}, \dots, \sqrt{q_r}\}$ is independent of the decomp.

Now suppose R is Noetherian. we want to show that any prime ideal of the form $\sqrt{I:a}$ is equal to $I:\hat{a}$ for some $\hat{a} \in R$

From above we know that $\sqrt{I:a} = \sqrt{q_{i_0}}$ for some i_0

$\therefore$ it is enough to show that for any i_0

$\sqrt{q_{i_0}} = I : \hat{a}$ for some $\hat{a} \in R$

$\therefore$ by the lemma 2 we have $(\sqrt{q_{i_0}})^n \subseteq q_{i_0}$ for some $n \geq 0$ (since R is Noetherian)

$\therefore$ $\exists n$ s.t

$$\left(\bigcap_{j\neq i_0} q_j\right)\cdot\left(\sqrt{q_{i_0}}\right)^n \subseteq \underbrace{\left(\bigcap_{j \neq i_0} q_j\right) q_{i_0}}_{\in q_{i_0} \text{ and } \in \bigcap_{j\neq i_0} q_j} \subseteq I$$

Fix the smallest n s.t $\left(\bigcap_{j\neq i_0} q_j\right)\cdot\left(\sqrt{q_{i_0}}\right)^n \subseteq I$

Pick some

$$\hat{a} \in \left(\bigcap_{j\neq i_0} q_j\right)\cdot\left(\sqrt{q_{i_0}}\right)^{n-1} - I$$

Then we have $\tilde{a}\sqrt{q_{i_0}} \subseteq \left(\bigcap_{j\neq i_0} q_j\right)\left(\sqrt{q_{i_0}}\right)^n \subseteq I$

$\therefore$ By the def $I:\tilde{a}$

$$\Rightarrow \sqrt{q_{i_0}} \subseteq I:\tilde{a}$$

Also note $\left(\bigcap_{j\neq i_0} q_j\right)\left(\sqrt{q_{i_0}}\right)^{n-1} \subseteq \bigcap_{j\neq i_0} q_j$

$\therefore$ $\hat{a} \in \bigcap_{j\neq i_0} q_j - I \Rightarrow \hat{a} \notin q_{i_0}$

$\therefore$ since $\tilde{a} \in \bigcap_{j\neq i_0} q_j$ and $\hat{a} \in q_{i_0}$

$$\Rightarrow \sqrt{I:\tilde{a}} = \sqrt{q_{i_0}}$$

But we showed $\sqrt{q_{i_0}} \subseteq I:\tilde{a} \subseteq \sqrt{I:\tilde{a}} = \sqrt{q_{i_0}}$

$\therefore \sqrt{q_{i_0}} = I:\hat{a}$ $\therefore$ for any i_0

we can find $\tilde{a}$ s.t. $\sqrt{q_{i_0}} = I:\tilde{a}$ $\blacksquare$